On finite W -algebras for Lie superalgebras in the regular case

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1. INTRODUCTION

- The *finite W -algebras* are certain associative algebras associated to a complex semi-simple Lie algebra $\mathfrak{g}$ and a nilpotent element $e \in \mathfrak{g}$.
- They are quantizations of Poisson algebras of functions on the Slodowy slice at e to the orbit $Ad(G)e$, where $\mathfrak{g} = Lie(G)$.
- Due to recent results of I. Losev, A. Premet and others, W -algebras play a very important role in description of primitive ideals.
- Finite W -algebras for semi-simple Lie algebras were introduced by A. Premet [13]. They have been studied by mathematicians and physicists: L. Fehér, I. Losev, V. Ginzburg, W. L. Gan, J. Brundan, S. Goodwin, W. Wang [1, 3, 6, 7, 8, 10, 15].

Let $\mathfrak{g}$ be a finite-dimensional semi-simple or reductive Lie algebra over $\mathbb{C}$,
 $(\cdot|\cdot)$ be a non-degenerate invariant symmetric bilinear form.

Definition 1.1. *Adjoint representation of $\mathfrak{g}$:*

$$ad(x)(y) = [x, y], \quad x, y \in \mathfrak{g}$$

Definition 1.2. $e \in \mathfrak{g}$ is **nilpotent**, if $ad(e)$ is a nilpotent endomorphism of $\mathfrak{g}$.

Example 1.3. $\mathfrak{g} = \mathfrak{gl}(n)$

$e \in \mathfrak{gl}(n)$ is nilpotent $\iff e$ is an $n \times n$ -matrix with eigenvalues zero.

Definition 1.4. A nilpotent element $e \in \mathfrak{g}$ is **regular nilpotent**, if $\mathfrak{g}^e = \text{Ker}(ad(e))$ attains the minimal dimension, which is equal to $\text{rank } \mathfrak{g}$.

Theorem 1.5. *Jacobson-Morozov [5].*

Associated to a nonzero nilpotent element $e \in \mathfrak{g}$, there always exists an $\mathfrak{sl}(2)$ -triple $\{e, h, f\}$ which satisfies

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

Proof. Induction on $\dim \mathfrak{g}$.

2. $\mathbb{Z}$ -GRADINGS

Definition 2.1. *A Dynkin $\mathbb{Z}$ -grading.*

Let $\mathfrak{sl}(2) = \langle e, h, f \rangle$. The eigenspace decomposition of the adjoint action

$$ad(h) : \mathfrak{g} \longrightarrow \mathfrak{g}$$

provides a $\mathbb{Z}$ -grading:

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j, \quad \mathfrak{g}_j = \{x \in \mathfrak{g} \mid ad(h)(x) = jx\}.$$

Properties:

- (1) $e \in \mathfrak{g}_2$,
- (2) $ad(e) : \mathfrak{g}_j \longrightarrow \mathfrak{g}_{j+2}$ is injective for $j \leq -1$,
- (3) $ad(e) : \mathfrak{g}_j \longrightarrow \mathfrak{g}_{j+2}$ is surjective for $j \geq -1$,
- (4) $\mathfrak{g}^e \subset \bigoplus_{j \geq 0} \mathfrak{g}_j$,
- (5) $(\mathfrak{g}_i | \mathfrak{g}_j) = 0$ unless $i + j = 0$,
- (6) $\dim \mathfrak{g}^e = \dim \mathfrak{g}_0 + \dim \mathfrak{g}_1$.

Definition 2.2. *A good $\mathbb{Z}$ -grading.*

*A $\mathbb{Z}$ -grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ for a semi-simple $\mathfrak{g}$ is called a **good $\mathbb{Z}$ -grading** for e , if it satisfies the conditions (1)-(3).*

For a reductive $\mathfrak{g}$, there is an additional condition: the center of $\mathfrak{g}$ is in $\mathfrak{g}_0$.

Remark 2.3. *Properties (4)-(6) remain to be valid for every good $\mathbb{Z}$ -grading of $\mathfrak{g}$.*

Remark 2.4.

Dynkin $\mathbb{Z}$ -grading $\implies$ Good $\mathbb{Z}$ -grading.

Good $\mathbb{Z}$ -grading $\not\implies$ Dynkin $\mathbb{Z}$ -grading.

3. DEFINITION OF FINITE W -ALGEBRAS

$\mathfrak{g}$ is a reductive Lie algebra,

$(\cdot|\cdot)$ is a non-degenerate invariant symmetric bilinear form,

e is a nilpotent element,

$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ is a good $\mathbb{Z}$ -grading for e ,

$\chi \in \mathfrak{g}^* \quad \chi(x) := (x|e) \quad \forall x \in \mathfrak{g}.$

Define a bilinear form on $\mathfrak{g}_{-1}$ as

$$(x, y) := ([x, y] | e) = \chi([x, y]) \quad \forall x, y \in \mathfrak{g}_{-1}$$

Remark 3.1. *The bilinear form on $\mathfrak{g}_{-1}$ is skew-symmetric and non-degenerate.*

Proof. The skew-symmetry follows by definition.

The non-degeneracy follows from the bijection

$$ad(e) : \mathfrak{g}_{-1} \longrightarrow \mathfrak{g}_1$$

and the identity

$$(x, y) = (x | [y, e]).$$

Hence $\dim \mathfrak{g}_{-1}$ is even.

Pick a Lagrangian (a maximal isotropic) subspace $\mathfrak{l}$ of $\mathfrak{g}_{-1}$ with respect to the form $(\cdot, \cdot)$.
Then $\dim \mathfrak{l} = \frac{1}{2} \dim \mathfrak{g}_{-1}$.

Let $\mathfrak{m} = (\oplus_{j \leq -2} \mathfrak{g}_j) \oplus \mathfrak{l}$.

The restriction of χ to $\mathfrak{m}$

$$\chi : \mathfrak{m} \longrightarrow \mathbb{C}$$

defines a one-dimensional representation $\mathbb{C}_\chi$ of $\mathfrak{m}$ thanks to the Lagrangian condition on $\mathfrak{l}$.

Let I_χ be the left ideal of $U(\mathfrak{g})$ generated by $a - \chi(a)$ for $a \in \mathfrak{m}$.

Definition 3.2. *The generalized Whittaker module is*

$$Q_\chi := U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi \cong U(\mathfrak{g})/I_\chi.$$

Definition 3.3. *The finite W -algebra associated to the nilpotent element e is*

$$W_\chi := \text{End}_{U(\mathfrak{g})}(Q_\chi)^{op}.$$

Remark 3.4. W_χ can be identified as the space of Whittaker vectors in $U(\mathfrak{g})/I_\chi$.

Let $\pi : U(\mathfrak{g}) \rightarrow U(\mathfrak{g})/I_\chi$ be the natural projection, and let $y \in U(\mathfrak{g})$.

$$W_\chi = (Q_\chi)^{ad\mathfrak{m}} = \{\pi(y) \in U(\mathfrak{g})/I_\chi \mid [a, y] \in I_\chi \quad \forall a \in \mathfrak{m}\}.$$

The multiplication is

$$\pi(y_1)\pi(y_2) = \pi(y_1y_2)$$

for $y_i \in U(\mathfrak{g})$ such that $[a, y_i] \in I_\chi \quad \forall a \in \mathfrak{m}$ and $i = 1, 2$.

Remark 3.5. The isoclasses of finite W -algebras do not depend on good $\mathbb{Z}$ -grading and Lagrangian subspace $\mathfrak{l}$ [3, 6].

Example 3.6. Let $e = 0$. Then $\chi = 0$, $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{m} = 0$,

$$Q_\chi = U(\mathfrak{g}), \quad W_\chi = U(\mathfrak{g}).$$

Theorem 3.7. B. Kostant, 1978 [9].

For a regular nilpotent element $e \in \mathfrak{g}$, $W_\chi \cong Z(\mathfrak{g})$, the center of $U(\mathfrak{g})$.

Definition 3.8. *Kazhdan filtration on W_χ .*

Let $\mathfrak{g}$ be a reductive Lie algebra with a Dynkin $\mathbb{Z}$ -grading: $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$,

$$\mathfrak{m} = \left(\bigoplus_{j \leq -2} \mathfrak{g}_j \right) \oplus \mathfrak{l}.$$

Let $\mathfrak{n} \subset \mathfrak{g}$ be an $\text{ad}(h)$ -invariant subspace such that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{n}$,

$\mathbb{C}\chi = \langle v \rangle$ be one-dimensional representation of $\mathfrak{m}$. Then

$$W_\chi = \{X \in U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m} \cong S(\mathfrak{n}) \mid aXv = \chi(a)Xv, \quad \forall a \in \mathfrak{m}\}.$$

For any $y \in \mathfrak{n}$, let $\text{wt}(y)$ be the weight of y with respect to $\text{ad}(h)$ and

$$\text{deg}(y) = \text{wt}(y) + 2.$$

The degree function deg induces a $\mathbb{Z}$ -grading on $S(\mathfrak{n})$. This grading defines a filtration on W_χ .

Theorem 3.9. *A. Premet [13].*

The associated graded algebra $Gr(W_\chi)$ is isomorphic to $S(\mathfrak{g}^e)$.

Idea of Proof.

Introduce the map

$$P : W_\chi \longrightarrow S(\mathfrak{g}^e).$$

For $X \in W_\chi \subset S(\mathfrak{n})$ the term $P(X)$ of highest degree and highest weight belongs to $S(\mathfrak{g}^e)$.

Example 3.10. *If $e = 0$, this is the PBW Theorem:*

$$Gr(U(\mathfrak{g})) \cong S(\mathfrak{g}).$$

4. THE CASE OF $\mathfrak{g} = \mathfrak{gl}(n)$

Form: $(a|b) = \text{tr}(ab)$

$$e = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ n-1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2(n-2) & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n-1 & 0 \end{pmatrix}$$

$h = \text{diag}(n-1, n-3, \dots, 3-n, 1-n)$.

e is a regular nilpotent element.

h defines an even Dynkin $\mathbb{Z}$ -grading of $\mathfrak{g}$ whose degrees on the elementary matrices E_{ij} are

$$\begin{pmatrix} 0 & 2 & 4 & 6 & \cdots & 2n-2 \\ -2 & 0 & 2 & 4 & \cdots & 2n-4 \\ -4 & -2 & 0 & 2 & \cdots & 2n-6 \\ -6 & -4 & -2 & 0 & \cdots & 2n-8 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 2-2n & \cdots & -6 & -4 & -2 & 0 \end{pmatrix}$$

$z = \text{diag}(1, \dots, 1)$ is the center of $\mathfrak{gl}(n)$.

$$\mathfrak{g}^e = \langle z, e, e^2, e^3, \dots, e^{n-1} \rangle, \quad \dim \mathfrak{g}^e = n$$

$$\mathfrak{m} = \bigoplus_{j=2}^n \mathfrak{g}_{2-2j}, \quad \chi(E_{i+1,i}) = 1, \quad \chi(E_{i+k,i}) = 0 \text{ if } k \geq 2$$

W_χ is a polynomial algebra generated by n elements:

$$\pi(z), \quad \pi(\Omega_2), \quad \pi(\Omega_3), \quad \dots, \quad \pi(\Omega_n)$$

Ω_k is the k -th Casimir element of $\mathfrak{gl}(n)$:

$$\Omega_k = \sum_{i_1, i_2, \dots, i_k} E_{i_1 i_2} E_{i_2 i_3} \dots E_{i_k i_1}$$

The generators of W_χ can be identified with elements of $\mathfrak{g}^e$:

$$\pi(z) \xrightarrow{P} z,$$

$$\frac{1}{k} \pi(\Omega_k) \xrightarrow{P} e^{k-1}, \quad \text{for } k = 2, \dots, n.$$

5. LIE SUPERALGEBRAS

Definition 5.1. *A Lie superalgebra is a $\mathbb{Z}_2$ -graded algebra*

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$$

with an operation $[\ , \]$ satisfying the following axioms:

1. super-anticommutativity:

$$[x, y] = -(-1)^{p(x)p(y)}[y, x] \quad \text{for } x, y \in \mathfrak{g},$$

2. super-Jacobi identity:

$$[x, [y, z]] = [[x, y], z] + (-1)^{p(x)p(y)}[y, [x, z]] \quad \text{for } x, y, z \in \mathfrak{g}.$$

Remark 5.2. $\mathfrak{g}_{\bar{0}}$ *is a Lie algebra,*

$$[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] \subseteq \mathfrak{g}_{\bar{1}} \implies \mathfrak{g}_{\bar{1}} \text{ is a module over } \mathfrak{g}_{\bar{0}},$$

$$[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subseteq \mathfrak{g}_{\bar{0}}.$$

General Linear Lie superalgebra

$$\mathfrak{g} = \mathfrak{gl}(m|n) = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}},$$

$$\mathfrak{g}_{\bar{0}} = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right) \mid A \text{ is a } m \times m \text{ matrix, } B \text{ is a } n \times n \text{ matrix} \right\},$$

$$\mathfrak{g}_{\bar{1}} = \left\{ \left(\begin{array}{c|c} 0 & C \\ \hline D & 0 \end{array} \right) \mid C \text{ is a } m \times n \text{ matrix, } D \text{ is a } n \times m \text{ matrix} \right\}.$$

Super-bracket:

$$[X, Y] = XY - (-1)^{p(X)p(Y)} YX \text{ for } X, Y \in \mathfrak{g}.$$

Super-trace:

$$str \left(\begin{array}{c|c} A & C \\ \hline D & B \end{array} \right) = tr(A) - tr(B).$$

Definition 5.3. *A Lie superalgebra $\mathfrak{g}$ is simple, if it is not abelian and the only $\mathbb{Z}_2$ -graded ideals of $\mathfrak{g}$ are $\{0\}$ and $\mathfrak{g}$.*

Special Linear Lie superalgebra

$$\mathfrak{sl}(m|n) = \{X \in \mathfrak{gl}(m|n) \mid \text{str} X = 0\}.$$

$\mathfrak{sl}(m|n)$ is simple $\iff m \neq n$.

If $m = n$, then $\mathfrak{sl}(n|n)/\langle 1_{2n} \rangle$ is simple .

6. FINITE W -ALGEBRAS FOR LIE SUPERALGEBRAS

joint work with V. Serganova (UC Berkeley)

- Finite W -algebras for Lie superalgebras were studied by mathematicians and physicists: C. Briot, E. Ragoucy, J. Brundan, J. Brown, S. Goodwin, W. Wang, L. Zhao [2, 4, 15, 16].

Let $\mathfrak{g}$ be a classical simple Lie superalgebra, i.e. $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra, and $\mathfrak{g}$ has an invariant symmetric bilinear form.

Let $e \in \mathfrak{g}_{\bar{0}}$ be an even nilpotent element, and we fix $\mathfrak{sl}(2) = \langle e, h, f \rangle$.

- The above definition of W_{χ} makes sense, however Theorem of Kostant does not hold in this case since W_{χ} must have a non-trivial odd part, and the center of $U(\mathfrak{g})$ is even.
- Kazhdan filtration on W_{χ} can be defined exactly as in the Lie algebra case.

Proposition 6.1. *$Gr(W_\chi)$ is supercommutative.*

Remark 6.2. *If $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is **even**, one can construct the similar map*

$$P : W_\chi \longrightarrow S(\mathfrak{g}^e)$$

by taking the monomials of the highest degree and the highest weight.

*If $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is **odd**, then there exists an odd element θ in $\mathfrak{g}_{-1} \cap \mathfrak{l}^\perp$ such that $\pi(\theta) \in W_\chi$ and $\pi(\theta)^2 = 1$.*

7. THE CASE OF $\mathfrak{g} = \mathfrak{sl}(1|n)$

Form: $(a|b) = -str(ab)$

$$e = \left(\begin{array}{c|cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right), \quad f = \left(\begin{array}{c|cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & n-1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2(n-2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n-1 & 0 \end{array} \right)$$

$$h = \text{diag}(0|n-1, n-3, \dots, 3-n, 1-n)$$

e is a regular nilpotent element.

Elementary matrices:

$$E_{ij} = \left(\begin{array}{c|ccccc} h_0 & \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_n \\ \hline \xi_1 & h_1 & e_1 & \cdots & \cdots & \cdots \\ \xi_2 & f_1 & h_2 & e_2 & \cdots & \cdots \\ \xi_3 & \cdots & f_2 & h_3 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & e_{n-1} \\ \xi_n & \cdots & \cdots & \cdots & f_{n-1} & h_n \end{array} \right)$$

h defines a Dynkin $\mathbb{Z}$ -grading of $\mathfrak{g}$ whose degrees on the elementary matrices E_{ij} are

$$\left(\begin{array}{c|cccccc} 0 & 1-n & 3-n & \cdots & n-3 & \mathbf{n}-1 \\ \hline \mathbf{n}-1 & 0 & 2 & 4 & \cdots & 2n-2 \\ n-3 & -2 & 0 & 4 & \cdots & 2n-4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 3-n & 4-2n & \cdots & \cdots & 0 & 2 \\ 1-n & 2-2n \cdots & \cdots & \cdots & -2 & 0 \end{array} \right)$$

$$\dim(\mathfrak{g}^e) = (n|2)$$

$c = \text{diag}(n|1, \cdots, 1)$ is a central elements of $\mathfrak{g}_{\bar{0}}$.

$$\mathfrak{g}^e = \langle e, e^2, \cdots, e^{n-1}, c \mid \xi_1, \mu_n \rangle,$$

$$\mathfrak{m} = \left(\bigoplus_{j \leq -2} \mathfrak{g}_j \right) \bigoplus \mathfrak{l},$$

if n is odd, then $\mathfrak{l} = 0$, if $n = 2k$, then $\mathfrak{l} = \langle \xi_{k+1} \rangle$.

$\mathfrak{m}$ is generated by $f_1, \dots, f_n; \mu_1, \dots, \mu_{k-1}, \xi_{k+1}, \dots, \xi_n$, if $n = 2k$,

and by $f_1, \dots, f_n; \mu_1, \dots, \mu_k, \xi_{k+2}, \dots, \xi_n$, if $n = 2k + 1$.

$$\chi(f_i) = 1 \quad \chi(\mu_i) = \chi(\xi_i) = 0.$$

$\mathfrak{n}$ is an $ad(h)$ -invariant subspace of $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{n}$.

$\mathbb{Z}$ -grading $\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1$ is consistent with $\mathbb{Z}_2$ -grading.

Fix an $ad(h)$ -homogeneous bases:

$B(\mathfrak{m}_{-1})$ of $\mathfrak{m} \cap \mathfrak{g}^{-1}$, $B(\mathfrak{m}_1)$ of $\mathfrak{m} \cap \mathfrak{g}^1$, $B(\mathfrak{n}_{-1})$ of $\mathfrak{n} \cap \mathfrak{g}^{-1}$, $B(\mathfrak{n}_1)$ of $\mathfrak{n} \cap \mathfrak{g}^1$.

Set

$$R_1 = \left(\prod_{x \in B(\mathfrak{m}_1)} x \right) \left(\prod_{y \in B(\mathfrak{n}_{-1})} y \right),$$

$$R_2 = \left(\prod_{y \in B(\mathfrak{m}_{-1})} y \right) \left(\prod_{x \in B(\mathfrak{n}_1)} x \right).$$

$\pi(R_1)$ and $\pi(R_2)$ are both Whittaker vectors, and hence $\pi(R_1), \pi(R_2) \in W_\chi$.

Remark 7.1. *Casimir elements for $\mathfrak{gl}(m|n)$:*

$$p(E_{ij}) = p(i) + p(j),$$

$$p(i) = \begin{cases} 0 & \text{if } 1 \leq i \leq m \\ 1 & \text{if } m+1 \leq i \leq m+n \end{cases}$$

$$\Omega_k = \sum_{i_1, i_2, \dots, i_k} (-1)^{p(i_2) + \dots + p(i_k)} E_{i_1 i_2} E_{i_2 i_3} \dots E_{i_k i_1}.$$

Even generators of W_χ : $\pi(\Omega_k)$, $k = 2, \dots, n$ and $\pi(c)$.

The generators of W_χ can be identified with elements of $\mathfrak{g}^e$:

$$(-1)^{k+1} \frac{1}{k} \pi(\Omega_k) \xrightarrow{P} e^{k-1}, \quad k = 2, \dots, n,$$

$$\pi(c) \xrightarrow{P} c,$$

$$\pi(R_1) \xrightarrow{P} \xi_1,$$

$$\pi(R_2) \xrightarrow{P} \mu_n.$$

Theorem 7.2. ([12]).

Let $\mathfrak{g}$ be a Lie superalgebra of Type I and defect one (i.e. $\mathfrak{g} = \mathfrak{sl}(1|n)$ or $\mathfrak{osp}(2|2n-2)$). Let n be the rank of $\mathfrak{g}_{\bar{0}}$, c be a central element of $\mathfrak{g}_{\bar{0}}$, and $\Omega_2, \dots, \Omega_n$ be the first $n-1$ Casimir elements in $Z(\mathfrak{g})$. Then W_χ is a finite extension of $\mathbb{C}[\pi(c), \pi(\Omega_2), \dots, \pi(\Omega_n)]$ with odd generators $\pi(R_1), \pi(R_2)$ and defining relations

$$\pi(R_1)^2 = \pi(R_2)^2 = 0, \quad [\pi(c), \pi(R_1)] = -\pi(R_1), \quad [\pi(c), \pi(R_2)] = \pi(R_2),$$

$$[\pi(\Omega_i), \pi(R_1)] = [\pi(\Omega_i), \pi(R_2)] = 0, \quad i = 2, \dots, n,$$

$$[\pi(R_1), \pi(R_2)] = \pi(\tilde{\Omega}),$$

where

$$\tilde{\Omega} = \prod_{y \in B(\mathfrak{g}^{-1})} \text{ad } y \left(\prod_{x \in B(\mathfrak{g}^1)} x \right)$$

is an element of $Z(\mathfrak{g})$.

In this case $W_\chi \cong U(\mathfrak{g}^e)$.

Remark 7.3.

- *The finite W -algebras for $\mathfrak{g} = \mathfrak{gl}(m|n)$ were described as certain truncations of a shifted version of the super-Yangian $Y(\mathfrak{gl}(1|1))$ by J. Brown, J. Brundan and S. Goodwin in 2012 [4].*
- *The connection between the finite W -algebra for $\mathfrak{gl}(m|n)$ and $Y(\mathfrak{gl}(1|1))$ was foreseen by C. Briot and E. Ragoucy in 2003 [2].*

8. THE CASE OF $\mathfrak{g} = \mathbf{Q}(\mathbf{n})$

$$Q(n) := \left\{ \left(\begin{array}{c|c} A & B \\ \hline B & A \end{array} \right) \mid A, B \text{ are } n \times n \text{ matrices} \right\}$$

Let $otr \left(\begin{array}{c|c} A & B \\ \hline B & A \end{array} \right) = tr B$.

Remark 8.1. $Q(n)$ has one-dimensional center $\langle z \rangle$, $z = 1_{2n}$.

Let $SQ(n) = \{X \in Q(n) \mid otr X = 0\}$.

The Lie superalgebra $SQ(n)/\langle z \rangle$ is simple.

Let $e_{i,j}$ and $f_{i,j}$ be standard bases in A and B respectively:

$$e_{i,j} = \left(\begin{array}{c|c} E_{ij} & 0 \\ \hline 0 & E_{ij} \end{array} \right), \quad f_{i,j} = \left(\begin{array}{c|c} 0 & E_{ij} \\ \hline E_{ij} & 0 \end{array} \right).$$

$Q(n)$ admits an **odd** nondegenerate $\mathfrak{g}$ -invariant super symmetric bilinear form

$$(x|y) := otr(xy) \text{ for } x, y \in \mathfrak{g}.$$

Let $\mathfrak{sl}(2) = \langle e, h, f \rangle$, where

$$e = \sum_{i=1}^{n-1} e_{i,i+1}, \quad h = \text{diag}(n-1, n-3, \dots, 3-n, 1-n), \quad f = \sum_{i=1}^{n-1} i(n-i)e_{i+1,i}.$$

e is a regular nilpotent element.

h defines an even Dynkin $\mathbb{Z}$ -grading of $\mathfrak{g}$ whose degrees on the elementary matrices are

$$\left(\begin{array}{cccc|cccc} 0 & 2 & \cdots & 2n-2 & 0 & 2 & \cdots & 2n-2 \\ -2 & 0 & \cdots & 2n-4 & -2 & 0 & \cdots & 2n-4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 2-2n & \cdots & \cdots & 0 & 2-2n & \cdots & \cdots & 0 \\ \hline 0 & 2 & \cdots & 2n-2 & 0 & 2 & \cdots & 2n-2 \\ -2 & 0 & \cdots & 2n-4 & -2 & 0 & \cdots & 2n-4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 2-2n & \cdots & \cdots & 0 & 2-2n & \cdots & \cdots & 0 \end{array} \right)$$

Replace $e = \sum_{i=1}^{n-1} e_{i,i+1}$ by $E = \sum_{i=1}^{n-1} f_{i,i+1} \Rightarrow E$ is odd.

Let $\chi \in \mathfrak{g}^*$ be defined by $\chi(x) = (x|E)$.

$$\mathfrak{g}^E = \{z, e, e^2, \dots, e^{n-1} \mid H_0, H_1, \dots, H_{n-1}\}, \quad \dim(\mathfrak{g}^E) = (n|n),$$

where $H_0 = \sum_{i=1}^n (-1)^{i+1} f_{i,i}$, $H_1 = \sum_{i=1}^{n-1} (-1)^i f_{i,i+1}$, $\dots$, $H_{n-1} = (-1)^{n+1} f_{1,n}$.

$$H_0 = \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ \hline 1 & & & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -1 & & 0 & & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & & 0 & 0 & \dots & 0 \end{array} \right)$$

$$\mathfrak{m} = \bigoplus_{j=2}^n \mathfrak{g}_{2-2j}$$

and it is generated by $e_{i+1,i}$ and $f_{i+1,i}$ where $i = 1, \dots, n-1$,

$$\chi(e_{i+1,i}) = 1, \quad \chi(f_{i+1,i}) = 0.$$

A. Sergeev defined by induction the elements $e_{i,j}^{(m)}$ and $f_{i,j}^{(m)}$ belonging to $U(Q(n))$ [14]:

$$e_{i,j}^{(m)} = \sum_{k=1}^n e_{i,k} e_{k,j}^{(m-1)} + (-1)^{m+1} \sum f_{i,k} f_{k,j}^{(m-1)},$$

$$f_{i,j}^{(m)} = \sum_{k=1}^n e_{i,k} f_{k,j}^{(m-1)} + (-1)^{m+1} \sum f_{i,k} e_{k,j}^{(m-1)}.$$

Then

$$[e_{i,j}, e_{k,l}^{(m)}] = \delta_{j,k} e_{i,l}^{(m)} - \delta_{i,l} e_{k,j}^{(m)}, \quad [e_{i,j}, f_{k,l}^{(m)}] = \delta_{j,k} f_{i,l}^{(m)} - \delta_{i,l} f_{k,j}^{(m)},$$

$$[f_{i,j}, e_{k,l}^{(m)}] = (-1)^{m+1} \delta_{j,k} f_{i,l}^{(m)} - \delta_{i,l} f_{k,j}^{(m)}, \quad [f_{i,j}, f_{k,l}^{(m)}] = (-1)^{m+1} \delta_{j,k} e_{i,l}^{(m)} + \delta_{i,l} e_{k,j}^{(m)}.$$

Proposition 8.2. *A. Sergeev [14]*

The elements $\sum_{i=1}^n e_{i,i}^{(2m+1)}$ generate $Z(Q(n))$.

Proposition 8.3. $\pi(e_{n,1}^{(m)})$ and $\pi(f_{n,1}^{(m)})$ are Whittaker vectors.

Theorem 8.4. W_χ has n even generators $\pi(e_{n,1}^{(n+k-1)})$ and n odd generators $\pi(f_{n,1}^{(n+k-1)})$, $k = 1, \dots, n$.

The generators of W_χ can be identified with elements of $\mathfrak{g}^E$:

$$\begin{array}{ll} \pi(e_{n,1}^{(n)}) \xrightarrow{P} z, & \pi(f_{n,1}^{(n)}) \xrightarrow{P} H_0, \\ \pi(e_{n,1}^{(n+1)}) \xrightarrow{P} e, & \pi(f_{n,1}^{(n+1)}) \xrightarrow{P} H_1, \\ \dots & \dots \end{array}$$

Corollary 8.5.

$$W_\chi \cong U(Q(n))^{adm}.$$

Let $\mathfrak{p} := \oplus_{j \geq 0} \mathfrak{g}_j$. Let $\mathfrak{f} = \langle e_{i,i}, f_{i,i} \mid i = 1, \dots, n \rangle$,

and let $\vartheta : U(\mathfrak{p}) \longrightarrow U(\mathfrak{f})$ be the Harish-Chandra homomorphism.

Proposition 8.6. *The Harish-Chandra homomorphism is injective.*

Denote

$$x_i = e_{i,i}, \quad \xi_i = (-1)^{i+1} f_{i,i}.$$

Theorem 8.7. *Under the Harish-Chandra homomorphism:*

$$\begin{aligned} \vartheta(\pi(e_{n,1}^{(n+k-1)})) &= [\sum_{i_1 \geq i_2 \geq \dots \geq i_k} (x_{i_1} + (-1)^{k+1} \xi_{i_1}) \dots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k})]_{\text{even}}, \\ \vartheta(\pi(f_{n,1}^{(n+k-1)})) &= [\sum_{i_1 \geq i_2 \geq \dots \geq i_k} (x_{i_1} + (-1)^{k+1} \xi_{i_1}) \dots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k})]_{\text{odd}}. \end{aligned}$$

Theorem 8.8.

$$\pi(e_{n,1}^{(n+1)}) = \pi\left(\frac{1}{2} \sum_{i=1}^n e_{i,i}^2 + \sum_{i=1}^{n-1} e_{i,i+1} + \sum_{i < j} (-1)^{i-j} f_{i,i} f_{j,j} + \frac{1}{2} z^2 - z\right).$$

One can define odd generators $\Phi_0, \dots, \Phi_{n-1}$ of W_χ as follows:

$$\Phi_0 = \pi(f_{n,1}^{(n)}) = \pi(H_0),$$

$$\Phi_1 = [\pi(e_{n,1}^{(n+1)}), \Phi_0],$$

$\dots$

$$\Phi_{n-1} = [\pi(e_{n,1}^{(n+1)}), \Phi_{n-2}].$$

Then

$$[\Phi_m, \Phi_p] = 0, \text{ if } m + p \text{ is odd,}$$

$$[\Phi_m, \Phi_p] \in Z(Q(n)), \text{ if } m + p \text{ is even.}$$

Lemma 8.9. *For odd m and p we have*

$$[\pi(e_{n,1}^{(n+m)}), \pi(e_{n,1}^{(n+p)})] = 0.$$

We set

$$z_i = \pi(e_{n,1}^{(n+i)}) \quad \text{for odd } i,$$

$$z_i = [\Phi_0, \Phi_i] \quad \text{for even } i.$$

Theorem 8.10 (P-S). *Elements $z_0, \dots, z_{n-1}$ are algebraically independent in W_χ . Together with $\Phi_0, \dots, \Phi_{n-1}$ they form a complete set of generators in W_χ .*

Conjecture 8.11. *In the case when $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is even, it is possible to find a set of generators of W_χ such that even generators commute, and the commutators of odd generators are in $Z(\mathfrak{g})$.*

9. Yangian of $Q(n)$

- For a finite-dimensional semi-simple Lie algebra $\mathfrak{g}$, the Yangian of $\mathfrak{g}$ is an infinite-dimensional Hopf algebra $Y(\mathfrak{g})$. It is a deformation of the universal enveloping algebra of the Lie algebra of polynomial currents of $\mathfrak{g}$.
- Super-Yangian $Y(Q(n))$ was studied by M. Nazarov and A. Sergeev [11].

$Y(Q(n))$ is the associative unital superalgebra over $\mathbb{C}$ with the countable set of generators $T_{i,j}^{(m)}$, where $m = 1, 2, \dots$ and $i, j = \pm 1, \pm 2, \dots, \pm n$.

The $\mathbb{Z}_2$ -grading of the algebra $Y(Q(n))$:

$$p(T_{i,j}^{(m)}) = p(i) + p(j), \text{ where } p(i) = 0 \text{ if } i > 0 \text{ and } p(i) = 1 \text{ if } i < 0.$$

Formal series in $Y(Q(n))[[u^{-1}]]$:

$$T_{i,j}(u) = \delta_{ij} \cdot 1 + T_{i,j}^{(1)}u^{-1} + T_{i,j}^{(2)}u^{-2} + \dots$$

The relations in $Y(Q(n))[[u^{-1}, v^{-1}]]$:

$$\begin{aligned} & (u^2 - v^2)[T_{i,j}(u), T_{k,l}(v)] \cdot (-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)} \\ &= (u + v)(T_{k,j}(u)T_{i,l}(v) - T_{k,j}(v)T_{i,l}(u)) \\ & - (u - v)(T_{-k,j}(u)T_{-i,l}(v) - T_{k,-j}(v)T_{i,-l}(u)) \cdot (-1)^{p(k)+p(l)}. \end{aligned} \quad (1)$$

We also have the relations

$$T_{i,j}(-u) = T_{-i,-j}(u). \quad (2)$$

Relations (1) and (2) are equivalent to the following defining relations:

$$\begin{aligned}
& ([T_{i,j}^{(m+1)}, T_{k,l}^{(r-1)}] - [T_{i,j}^{(m-1)}, T_{k,l}^{(r+1)}]) \cdot (-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)} = \\
& T_{k,j}^{(m)} T_{i,l}^{(r-1)} + T_{k,j}^{(m-1)} T_{i,l}^{(r)} - T_{k,j}^{(r-1)} T_{i,l}^{(m)} - T_{k,j}^{(r)} T_{i,l}^{(m-1)} \\
& + (-1)^{p(k)+p(l)} (-T_{-k,j}^{(m)} T_{-i,l}^{(r-1)} + T_{-k,j}^{(m-1)} T_{-i,l}^{(r)} + T_{k,-j}^{(r-1)} T_{i,-l}^{(m)} - T_{k,-j}^{(r)} T_{i,-l}^{(m-1)}), \quad (1')
\end{aligned}$$

$$T_{-i,-j}^{(m)} = (-1)^m T_{i,j}^{(m)}, \quad (2')$$

where $m, r = 1, \dots$ and $T_{ij}^{(0)} = \delta_{ij}$.

Theorem 9.1 (P-S). *There exists a surjective homomorphism:*

$$\varphi : Y(Q(1)) \longrightarrow W_\chi$$

defined as follows:

$$\varphi(T_{1,1}^{(k)}) = (-1)^k \pi(e_{n,1}^{(n+k-1)}), \quad \varphi(T_{-1,1}^{(k)}) = (-1)^k \pi(f_{n,1}^{(n+k-1)}), \quad \text{for } k = 1, 2, \dots$$

10. THE CASE OF $\mathfrak{g} = \Gamma(\sigma_1, \sigma_2, \sigma_3)$

$\sigma_1, \sigma_2, \sigma_3$ are complex numbers, $\sigma_1 + \sigma_2 + \sigma_3 = 0$.

$$\Gamma(\sigma_1, \sigma_2, \sigma_3) = \Gamma_{\bar{0}} \oplus \Gamma_{\bar{1}},$$

$$\Gamma_{\bar{0}} = \mathfrak{sl}(2)_{\bar{1}} \oplus \mathfrak{sl}(2)_{\bar{2}} \oplus \mathfrak{sl}(2)_{\bar{3}},$$

$$\Gamma_{\bar{1}} = V_1 \otimes V_2 \otimes V_3.$$

$$\mathfrak{sl}(2)_i = \langle X_i, H_i, Y_i \rangle, \quad V_i = \langle e_i, f_i \rangle, \quad i = 1, 2, 3.$$

$P_i : V_i \times V_i \rightarrow \mathfrak{sl}(2)_i$ is $\mathfrak{sl}(2)_i$ -invariant bilinear mapping:

$$P_i(e_i, e_i) = 2X_i, \quad P_i(f_i, f_i) = -2Y_i, \quad P_i(e_i, f_i) = P_i(f_i, e_i) = -H_i.$$

ψ_i is a non-degenerate skew-symmetric form on V_i :

$$\psi_i(e_i, f_i) = 1.$$

$[\Gamma_{\bar{0}}, \Gamma_{\bar{1}}]$ is the natural representation.

$[\Gamma_{\bar{1}}, \Gamma_{\bar{1}}]$ is given by

$$[x_1 \otimes x_2 \otimes x_3, y_1 \otimes y_2 \otimes y_3] = \sigma_1 \psi_2(x_2, y_2) \psi_3(x_3, y_3) P_1(x_1, y_1) + \\ \sigma_2 \psi_1(x_1, y_1) \psi_3(x_3, y_3) P_2(x_2, y_2) + \sigma_3 \psi_1(x_1, y_1) \psi_2(x_2, y_2) P_3(x_3, y_3).$$

Remark 10.1. *The superalgebra $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ is simple $\iff \sigma_i \neq 0$ for $i = 1, 2, 3$.*

$\Gamma(\sigma_1, \sigma_2, \sigma_3) \cong \Gamma(\sigma'_1, \sigma'_2, \sigma'_3) \iff$ the sets $\{\sigma'_i\}$ and $\{\sigma_i\}$ are obtained from each other by a permutation and multiplication of all elements of one set by a nonzero complex number.

$\Gamma(\sigma_1, \sigma_2, \sigma_3)$ is a one-parameter family of deformations of $osp(4|2)$.

$\Gamma(1, -1 - \alpha, \alpha) \cong D(2, 1; \alpha)$, where $\alpha \neq 0, -1$.

Consider the form on $\mathfrak{g}$ is determined by

$$(X_i, Y_i) = \frac{1}{\sigma_i}.$$

$\mathfrak{sl}(2) = \langle e, h, f \rangle$, where

$$e = X_1 + X_2 + X_3, \quad h = H_1 + H_2 + H_3, \quad f = Y_1 + Y_2 + Y_3.$$

e is a regular nilpotent element, h defines a Dynkin $\mathbb{Z}$ -grading of $\mathfrak{g}$:

$$\mathfrak{g} = \bigoplus_{j=-3}^3 \mathfrak{g}_j, \text{ where}$$

$$\begin{aligned} \mathfrak{g}_3 &= \langle e_1 \otimes e_2 \otimes e_3 \rangle, & \mathfrak{g}_2 &= \langle X_1, X_2, X_3 \rangle, \\ \mathfrak{g}_1 &= \langle e_1 \otimes e_2 \otimes f_3, e_1 \otimes f_2 \otimes e_3, f_1 \otimes e_2 \otimes e_3 \rangle, \\ \mathfrak{g}_0 &= \langle H_1, H_2, H_3 \rangle, & \mathfrak{g}_{-1} &= \langle e_1 \otimes f_2 \otimes f_3, f_1 \otimes e_2 \otimes f_3, e_1 \otimes e_2 \otimes f_3 \rangle, \\ \mathfrak{g}_{-2} &= \langle Y_1, Y_2, Y_3 \rangle, & \mathfrak{g}_{-3} &= \langle f_1 \otimes f_2 \otimes f_3 \rangle. \end{aligned}$$

$$\dim(\mathfrak{g}^e) = (3|3).$$

$$(\mathfrak{g}^e)_{\bar{0}} = \langle X_1, X_2, X_3 \rangle,$$

$$(\mathfrak{g}^e)_{\bar{1}} = \langle e_1 \otimes f_2 \otimes e_3 - e_1 \otimes e_2 \otimes f_3, f_1 \otimes e_2 \otimes e_3 - e_1 \otimes e_2 \otimes f_3, e_1 \otimes e_2 \otimes e_3 \rangle.$$

$$\mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{l}, \quad \mathfrak{l} = \langle e_1 \otimes f_2 \otimes f_3 \rangle.$$

$\mathfrak{m}$ is generated by Y_1, Y_2, Y_3 and $e_1 \otimes f_2 \otimes f_3$.

$$\chi(Y_i) = \frac{1}{\sigma_i}, \quad \chi(e_1 \otimes f_2 \otimes f_3) = 0.$$

$$\theta = f_1 \otimes e_2 \otimes f_3 - f_1 \otimes f_2 \otimes e_3 \in \mathfrak{g}_{-1} \cap \mathfrak{l}^\perp, \quad \pi(\theta) \in W_\chi, \quad \pi(\theta)^2 = -2.$$

Even generators of W_χ :

$$\mathbf{C}_1 = \pi(2X_1 + \sigma_1(\frac{H_1^2}{2} - H_1)),$$

$$\mathbf{C}_2 = \pi(2X_2 + \sigma_2\frac{H_2^2}{2} + (f_1 \otimes e_2 \otimes f_3)(e_1 \otimes e_2 \otimes f_3)),$$

$$\mathbf{C}_3 = \pi(2X_3 + \sigma_3\frac{H_3^2}{2} + (f_1 \otimes f_2 \otimes e_3)(e_1 \otimes f_2 \otimes e_3)).$$

Odd generators of W_χ :

$$\mathbf{R}_1 = \pi(2(e_1 \otimes f_2 \otimes e_3 - e_1 \otimes e_2 \otimes f_3) + \sigma_1 H_1(f_1 \otimes e_2 \otimes f_3 - f_1 \otimes f_2 \otimes e_3)),$$

$$\mathbf{R}_2 = \pi(2(f_1 \otimes e_2 \otimes e_3 - e_1 \otimes e_2 \otimes f_3) + (\sigma_1 H_1 - \sigma_3 H_3)(f_1 \otimes e_2 \otimes f_3) - \sigma_2 H_2(f_1 \otimes f_2 \otimes e_3)),$$

$$\begin{aligned} \mathbf{R}_3 = & \pi(4(e_1 \otimes e_2 \otimes e_3) - \sigma_1 H_1 \mathbf{R}_2 - 4\sigma_1(f_1 \otimes e_2 \otimes f_3)X_1 \\ & - 2(\sigma_1 H_1(e_1 \otimes e_2 \otimes f_3) + \sigma_2 H_2(e_1 \otimes f_2 \otimes e_3) + \sigma_3 H_3(e_1 \otimes e_2 \otimes f_3))), \end{aligned}$$

and $\pi(\theta)$.

$$\pi(\Omega) = C_1 + C_2 + C_3 - \frac{1}{2}R_1\pi(\theta),$$

where Ω is the Casimir element of $\mathfrak{g}$.

Theorem 10.2 (P-S). W_χ is generated by even elements $\pi(\Omega)$, C_1 and C_2 , and odd element $\pi(\theta)$. The relations are

$$[C_1, C_2] = 0, \quad [\pi(\theta), C_i] = R_i \mp \frac{\sigma_i}{2}\pi(\theta), \quad i = 1, 2,$$

$$[C_2, R_1] = -\frac{\sigma_2}{2}R_1 + R_3, \quad [C_1, R_2] = \frac{\sigma_1}{2}R_2 + R_3,$$

$$[R_i, R_i] = 8\sigma_i C_i - 2\sigma_i R_i \pi(\theta), \quad i = 1, 2,$$

$$[R_1, R_2] = -4(\sigma_1 C_2 + \sigma_2 C_1 + \sigma_3 \pi(\Omega)) + (\sigma_1 R_2 + \sigma_2 R_1) \pi(\theta),$$

$$[R_i, \pi(\theta)] = \mp 2\sigma_i, \quad i = 1, 2, \quad [\pi(\theta), \pi(\theta)] = -4,$$

$$[\pi(\Omega), \pi(\theta)] = 0, \quad [\pi(\Omega), C_i] = 0, \quad i = 1, 2, \quad [\pi(\Omega), R_i] = 0, \quad i = 1, 2, 3.$$

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