Nonlinear Mathematical Physics 1996, V.3, N 3–4, 372–378.

Symmetries of the Relativistic Two-Particle Model with Scalar-Vector Interaction

Askold DUVIRYAK

Institute for Condensed Matter Physics
of the Ukrainian National Academy of Sciences,
1 Svientsitskii Street, Lviv, 290011, Ukraïna

Abstract

A relativistic two-particle model with superposition of time-asymmetric scalar and vector interactions is proposed and its symmetries are considered. It is shown that first integrals of motion satisfy nonlinear Poisson-bracket relations which include the Poincaré algebra and one of the algebras *so(1,3)*, *so(4)* or *e(3)*.

Introduction

It is well-known that a nonrelativistic Galilei-invariant two-body system with Coulomb interaction possesses additional internal symmetries. These symmetries provide particle trajectories to be closed curves and lead to existence of an additional integral of motion. It is the Runge-Lentz vector. In framework of the Hamiltonian formalism, the components of this vector together with the internal angular momentum generate the algebra $O(4)$ or $O(1,3)$, when energy is negative or positive.

The Poincaré-invariant analog of such a system was proposed and its symmetries were studied by Droz-Vincent and Nurkowski [1] within the relativistic predictive mechanics. Their model is considered as a naïve exactly solvable relativistic generalization of a Coulombian two-body system rather than a description of some realistic interaction.

Nevertheless, the internal symmetries can occur in some physically interpretable relativistic two-body models. Here such a model is constructed in framework of relativistic light-cone mechanics [2–4]. Namely, the two-body system with equal–weighed superposition of vector and scalar time-asymmetric interactions naturally arises under transition from the Fokker scalar-vector model [5] (Section 1) toward its manifestly covariant Hamiltonian formulation (Section 2) . In Section 3, existence of the relevant relativistic Runge-Lentz vector is shown and its Poisson bracket relations are calculated. In Section 4, these results are reformulated for more evidence in framework of the Bakamjian-Thomas model [6]. The Section 5 shows how to obtain the relative and particle trajectories using the Runge-Lentz vector instead of integration of the equations of motion.

1 Time-Asymmetric Model With Scalar-Vector Interaction

We start with the Fokker action integral for a two-particle system with an arbitrary superposition of scalar and vector interactions [5]:

$$I = \sum_{a=1}^{2} m_a \int d\tau_a \sqrt{\dot{x}_a^2} \quad + \quad \int\int d\tau_1 d\tau_2 (\alpha \dot{x}_1 \cdot \dot{x}_2 + \beta \sqrt{\dot{x}_1^2}\sqrt{\dot{x}_2^2}) G(x). \tag{1}$$

Here m_a $(a = 1, 2)$ is the rest mass of the a-th particle; $x^a_\mu(\tau_a)$ $(\mu = 0...3)$ are the covariant coordinates of the a-th particle with the world line in the Minkowski space parametrized by an arbitrary parameter τ_a; $x^\mu \equiv x^1_\mu - x^2_\mu$; $\dot{x}^\mu_a \equiv dx^\mu_a/d\tau_a$; α and β are the coupling constants of vector and scalar interactions, respectively; $G(x) = \delta(x^2)$ is symmetrical Green's function of the d'Alambert equation. We choose the time-like Minkowski metrics, i.e., $\| \eta_{\mu\nu} \| = diag(+, -, -, -)$, and put the light speed to be unit.

The action (1) describes the system of the infinite number of degrees of freedom because the equations of motion are difference-differential. The analysis of their solutions is a complicated task.

The possible way to restrict the degrees of freedom to the finite number which is the same as in a nonrelativistic case is to replace symmetric Green's function in the right-hand side (r.h.s.) of Eq.(1) by the retarded (advanced) one [7, 2–4]

$$G(x) = 2\Theta(\eta x^0)\delta(x^2), \quad \eta = \pm 1. \tag{2}$$

Such a choice corresponds to the model in which particles interact in the following way: advanced fields of the first particle act on the second and retarded fields of the second particle act on the first.

The Green's function (2) does not vanish only on that pairs of points of particle world lines which satisfy the following condition:

$$x^2 = 0, \quad \eta x^0 > 0. \tag{3}$$

This condition looks like the light-cone equation and permits to reduce the Fokker integral (1)–(2) to the single-time action:

$$I = \int d\tau (L + \lambda x^2), \tag{4}$$

where the Lagrangian multiplier λ takes into account the light-cone condition (3) as the holonomic constraint (an unkeeping constraint $\eta x^0 > 0$ is implied also), and the Lagrangian function has the form:

$$L = \sum_{a=1}^{2} m_a \sqrt{\dot{x}_a^2} \quad + \quad \frac{\alpha \dot{x}_1 \cdot \dot{x}_2 + \beta \sqrt{\dot{x}_1^2}\sqrt{\dot{x}_2^2}}{\eta \dot{y} \cdot x}, \tag{5}$$

where $y^\mu \equiv \frac{1}{2}(x^\mu_1 + x^\mu_2)$.

2 Manifest-Covariant Hamiltonian Formalism

Due to the manifest covariance of the Lagrangian formulation of the model, the transition to the Hamiltonian formalism gives a 16-dimensional phase space with Poisson brackets $[...,...]$ in terms of covariant coordinates y^μ, x^μ and conjugated momenta defined in usual manner, i.e.,

$$P_\mu = \partial L/\partial \dot{y}^\mu, \qquad w_\mu = \partial L/\partial \dot{x}^\mu. \tag{6}$$

Since the Lagrangian (5) and constraint (3) are Poincaré-invariant, ten Noether's integrals of motion exist. They are the total momentum of the system P_μ (6) and the total angular momentum tensor

$$J_{\mu\nu} = y_\mu P_\nu - y_\nu P_\mu + x_\mu w_\nu - x_\nu w_\mu. \tag{7}$$

Within the Hamiltonian description these P_μ and $J_{\mu\nu}$ satisfy the canonical relations of the Poincaré algebra:

$$\begin{aligned} &[P_\mu, P_\nu] = 0, \qquad [P_\mu, J_{\lambda\sigma}] = -\eta_{\mu\lambda}P_\sigma + \eta_{\mu\sigma}P_\lambda, \\ &[J_{\mu\nu}, J_{\lambda\sigma}] = \eta_{\mu\lambda}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\lambda} - \eta_{\mu\sigma}J_{\nu\lambda} - \eta_{\nu\lambda}J_{\mu\sigma}. \end{aligned} \tag{8}$$

By virtue of parametric invariance of the action (4), the Lagrangian (5) is singular. Thus, the Hamiltonian vanishes, i.e., $H = \dot{y}\cdot P + \dot{x}\cdot w - L = 0$, and relations (6) are not invertible. As a consequence, the dynamical so-called mass-shell constraint appears which forms together with the light-cone constraint (3) the pair of first class ones. The mass-shell constraint has the following form:

$$\phi = \phi_f + \phi_{int} = 0, \tag{9}$$

where

$$\phi_f = \frac{1}{4}P^2 - \frac{1}{2}(m_1^2 + m_2^2) + (m_1^2 - m_2^2)\frac{v\cdot x}{P\cdot x} + v^2 \tag{10}$$

is the free-particle term, and

$$\begin{aligned} \phi_{int} \;=\; & -\frac{\alpha(P^2 - m_1^2 - m_2^2) + 2\beta m_1 m_2}{\eta P\cdot x} \;+ \\ & (\alpha^2 - \beta^2)\frac{(b_1-\alpha)m_2^2 + (b_2-\alpha)m_1^2 + 2\beta m_1 m_2}{\eta P\cdot x((b_1-\alpha)(b_2-\alpha) - \beta^2)} \end{aligned} \tag{11}$$

describes the interaction. Here the following notations are used:

$$v^\mu \equiv P^\nu \Omega_{\nu\mu}/P\cdot x, \quad \Omega_{\mu\nu} = x_\mu w_\nu - x_\nu w_\mu; \quad v\cdot P \equiv 0, \tag{12}$$

$$b_a \equiv \eta\Big(\frac{1}{2}P\cdot x + (-)^{\bar a} v\cdot x\Big), \qquad a = 1,2; \quad \bar a \equiv 3 - a. \tag{13}$$

The complicated structure of the mass-shell constraint promises interesting results of a mechanical analysis of the model. This general case, however, is not a subject of the present work. The examples of purely vector and scalar time-asymmetric interactions have been studied in details in [4].

In the following special case $\beta = \kappa\alpha$, $\kappa = \pm 1$, the second term of ϕ_{int} (11) vanishes and the mass-shell constraint becomes simpler to a great extent. For such an equal-weighed superposition of scalar and vector interactions, one can expect the appearance of internal symmetry. Further we take this case for our consideration and look for a relevant relativistic analog of the Runge-Lentz vector.

3 Relativistic Runge-Lentz Vector

It is convenient to simplify the free-particle term ϕ_f (10) of the mass-shell constraint, which cumbersome form obscures a following treatment of the model and is caused by a descriptional rather than dynamical reason. Let us perform the canonical transformation $(y^\mu,\ \ P_\mu,\ \ x^\mu,\ \ w_\mu) \longmapsto (z^\mu,\ \ P_\mu,\ \ x^\mu,\ \ q_\mu)$ using the generator function:

$$F(y,P,x,q) = y\cdot P + x\cdot q + \frac{m_1^2 - m_2^2}{2P^2} P\cdot x. \tag{14}$$

The new variables z^μ and q_μ are related with the original ones as follows:

$$w_\mu = \frac{\partial F}{\partial x^\mu} = q_\mu + \frac{m_1^2 - m_2^2}{2P^2}P_\mu, \quad z^\mu = \frac{\partial F}{\partial P_\mu} = y^\mu + \frac{m_1^2 - m_2^2}{2P^2}\Big(x^\mu - 2\frac{P\cdot x}{P^2}P^\mu\Big), \tag{15}$$

whereas the variables x^μ and P_μ remain unchanged. In terms of new variables the mass–shell constraint takes the form:

$$\phi = \frac{1}{4}P^2 - \frac{1}{2}(m_1^2 + m_2^2) + \frac{(m_1^2 - m_2^2)^2}{4P^2} + u^2 - \frac{\alpha(P^2 - (m_1 - \kappa m_2)^2)}{\eta P\cdot x} = 0, \tag{16}$$

where

$$u^\mu \equiv P^\nu \Xi_{\nu\mu}/P\cdot x, \quad \Xi_{\mu\nu} = x_\mu q_\nu - x_\nu q_\mu; \quad u\cdot P \equiv 0. \tag{17}$$

Let us define the relativistic analog of the Runge-Lentz vector as follows:

$$R_\mu = \Pi_\mu^\nu\Big(u^\lambda \Xi_{\lambda\nu} + \frac{\alpha(P^2 - (m_1 - \kappa m_2)^2)}{2\eta P\cdot x}x_\nu\Big), \tag{18}$$

where $\Pi_\mu^\nu \equiv \delta_\mu^\nu - P_\mu P^\nu/P^2$. It is easy to examine that R_μ is the integral of motion, i.e.,

$$[R_\mu, \phi] \approx 0, \qquad [R_\mu, x^2] = 0 \tag{19}$$

and satisfies the relations:

$$[R_\mu, P_\nu] = 0, \qquad [R_\mu, J_{\lambda\sigma}] = -\eta_{\mu\lambda}R_\sigma + \eta_{\mu\sigma}R_\lambda, \tag{20}$$

$$[R_\mu, R_\nu] \approx \Big(\frac{1}{4}P^2 - \frac{1}{2}(m_1^2 + m_2^2) + \frac{(m_1^2 - m_2^2)^2}{4P^2}\Big)\Pi_\mu^\lambda \Pi_\nu^\sigma J_{\lambda\sigma}, \tag{21}$$

where the Dirac's symbol $\approx$ denotes week equality.

The relations (20)–(21) are similar to those obtained for the Runge-Lentz vector of simple relativistic Coulomb model and discussed in ref.[1]. Here we note that these relations together with the relations (8) do not form algebra. The reason resides in essential nonlinearity of r.h.s. of Eq.(21). Nevertheless, the given system really possesses internal symmetries. In order to single out ones, let us reformulate the present time-asymmetrical model in the framework of the well-known Bakamjian-Thomas (BT) model [6].

4 The Description in Framework of the Bakamjian–Thomas Model

The BT model is based on 12-dimensional phase space with canonical variables Q^i, P_i, r^i, k_i ($i = 1, 2, 3$) and correspondent Poisson brackets $\{..., ...\}$. The essential, i.e., internal dynamics of the BT model is determined by the total mass of a system $|P| = M(\mathbf{r}, \mathbf{k})$ being the integral of motion. In our case this scalar function of 3-vector arguments $\mathbf{r}, \mathbf{k}$ is determined implicitly by means of the mass-shell equation being the consequent of the mass-shell constraint (16):

$$h(M) - \mathbf{k}^2 - 2g(M)/r = 0; \tag{22}$$

here $r \equiv |\mathbf{r}|$ and

$$h(M) \equiv \frac{1}{4M^2}\Big(M^2 - (m_1 + m_2)^2\Big)\Big(M^2 - (m_1 - m_2)^2\Big), \tag{23}$$

$$g(M) \equiv \frac{\alpha}{2M}\Big(M^2 - (m_1 - \kappa m_2)^2\Big). \tag{24}$$

By virtue of Poincaré-invariance of the BT-description it is sufficiently to choose the center-of-mass (CM) reference frame by putting $\mathbf{P} = \mathbf{0}$, $\mathbf{Q} = \mathbf{0}$. Then the energy of the system P_0 is M and other integrals of motion (7), (18) become as follows: $R_0 = 0$, $J_{i0} = 0$; $S_i \equiv \frac{1}{2}\varepsilon_i^{jk} J_{jk}$ and R_i form the 3-vectors $\mathbf{S} = \mathbf{r} \times \mathbf{k}$ and

$$\mathbf{R} = \mathbf{k} \times \mathbf{S} + g(M)\mathbf{r}/r. \tag{25}$$

Besides, the BT-description is supplemented by the relations of original covariant and canonical variables. In the CM reference frame the covariant particle positions are the following functions of the canonical variables [2,3]:

$$\mathbf{x}_a = (-)^{\bar{a}}\epsilon_{\bar{a}}\mathbf{r} + \eta r\mathbf{k}/M, \qquad a = 1, 2; \quad \bar{a} \equiv 3 - a, \tag{26}$$

where

$$\epsilon_a \equiv \frac{1}{2}\Big(1 + \frac{m_a^2 - m_{\bar{a}}^2}{M^2}\Big). \tag{27}$$

Especially, the relative position vector $\mathbf{x} \equiv \mathbf{x_1} - \mathbf{x_2} = \mathbf{r}$.

The Poisson bracket relations for the internal angular momentum (spin) of the system $\mathbf{S}$ and the Runge-Lentz vector $\mathbf{R}$ are similar to nonrelativistic ones for the Coulomb problem:

$$\{S_i, S_j\} = \varepsilon_{ij}^k S_k, \quad \{R_i, S_j\} = \varepsilon_{ij}^k R_k, \quad \{R_i, R_j\} = -h(M)\varepsilon_{ij}^k S_k. \tag{28}$$

Indeed, when $h(M) = 0$, Eq.(28) are the relations for generators of the Euclidean group $E(3)$. In case $h(M) \neq 0$, the S_i and the normalized $\hat{R}_i \equiv R_i/\sqrt{|h|}$ generate the group $SO(4)$, when $h(M) < 0$, and the group $SO(1,3)$, when $h(M) > 0$. Taking into account Eq.(23), one obtains the following cases for the algebras of internal symmetries:

$so(4)$ for $|m_1 - m_2| < M < m_1 + m_2$,
$e(3)$ for $M = |m_1 - m_2|$ and $M = m_1 + m_2$,
$so(1,3)$ for $0 < M < |m_1 - m_2|$ and $M > m_1 + m_2$.

5 Relative and Particle Trajectories Without Integration

The existence of the Runge-Lentz vector makes it possible to obtain both the relative and particle trajectories traced by vectors $\mathbf{r}$ and $\mathbf{x}_a$, respectively, without integration. First we note that these trajectories are flat curves placed on the plane orthogonal to the spin of the system, i.e., $\mathbf{r}\cdot\mathbf{S} = \mathbf{x}_a\cdot\mathbf{S} = 0$. The $\mathbf{R}$ lies on the same plane, i.e., $\mathbf{R}\cdot\mathbf{S} = 0$. Multiplying Eq.(25) by $\mathbf{r}$, one can obtain the relation:

$$\mathbf{R}\cdot\mathbf{r} = S^2 + gr, \tag{29}$$

where $S = |\mathbf{S}|$. Let φ be the angle between $\mathbf{R}$ and $\mathbf{r}$, i.e., $\mathbf{R}\cdot\mathbf{r} = Rr\cos\varphi$, where $R \equiv |\mathbf{R}|$ can be calculated by squaring Eq.(25):

$$\mathbf{R}^2 = hS^2 + g^2. \tag{30}$$

Now using Eqs.(23), (24) and (30), Eq.(29) can be reduced to the canonical equation of a conic section

$$p/r = e\cos\varphi - \mathrm{sgn} g \tag{31}$$

with the parameter p and the eccentricity e written down as follows:

$$p = \frac{S^2}{|g|} = \frac{2MS^2}{|\alpha||M^2 - (m_1 - \kappa m_2)^2|}, \qquad e = \frac{R}{|g|} = \sqrt{1 + \frac{S^2}{\alpha^2}\frac{M^2 - (m_1 + \kappa m_2)^2}{M^2 - (m_1 - \kappa m_2)^2}}. \tag{32}$$

The looking for equations of particle trajectories is a more complicated task. Let us define the vectors:

$$\mathbf{r}_a \equiv \mathbf{x}_a - \delta_a\mathbf{R}, \tag{33}$$

where δ_a are some functions of integrals of motion. Taking into account Eqs.(25)–(26), one can obtain the relations:

$$\mathbf{R}\cdot\mathbf{r}_a = (-)^{\bar{a}} gY_a - \delta_a R^2 + (-)^{\bar{a}}\epsilon_{\bar{a}} S^2, \tag{34}$$

$$\mathbf{r}_a^2 = Y_a^2 - 2(-)^{\bar{a}}\delta_a gY_a + \delta_a^2 R^2 - 2(-)^{\bar{a}}\delta_a\epsilon_{\bar{a}} S^2 + S^2/M^2. \tag{35}$$

We note that all the quantities in r.h.s. of Eqs.(34)–(35) except the

$$Y_a \equiv \epsilon_{\bar{a}} r + (-)^{\bar{a}}\eta\mathbf{r}\cdot\mathbf{k}/M \tag{36}$$

are constants of motion. Let us require for the square polynomials in r.h.s. of Eq.(35) to be linear binomials squared. Then the relevant discriminants must vanish:

$$D_a = -4S^2(h\delta_a^2 - 2(-)^{\bar{a}}\epsilon_{\bar{a}}\delta_a + 1/M^2) = 0. \tag{37}$$

The treatment of the conditions (37) as the square equations for δ_a and the choice of its solutions with less absolute values give the following expressions:

$$\delta_a = \frac{2(-)^{\bar{a}}}{(M + m_{\bar{a}})^2 - m_a^2}. \tag{38}$$

Just now Eq.(35) can be put into the equivalent form:

$$r_a \equiv |\mathbf{r}_a| = Y_a - (-)^{\bar{a}}\delta_a g. \qquad (39)$$

Eliminating the Y_a from r.h.s. of Eqs.(34), (39), one can obtain the relations:

$$(-)^{\bar{a}}\mathbf{R}\cdot\mathbf{r}_a = g r_a + \frac{m_{\bar{a}}}{M}S^2, \qquad (40)$$

which are similar to Eq.(29) and hence can be put into the form:

$$p_a/r_a = e\cos\varphi_a - \mathrm{sgn} g, \qquad (41)$$

where φ_a are the angles between $(-)^{\bar{a}}\mathbf{R}$ and $\mathbf{r}_a$. The equations (41) describe the particle trajectories being conic sections of the same shapes as the relative trajectory, i.e., with the same eccentricity e (32) but with another parameters $p_a = \frac{m_{\bar{a}}}{M}p$. The foci of these conic sections are shifted with respect to the center of mass by vectors $\delta_a\mathbf{R}$. On the contrary, the nonrelativistic particle trajectories have a common focus which is located in the center of mass.

Conclusion

The idea that internal symmetries are typical of nonrelativistic mechanics only is widespread among physicists. Indeed, the perihelion advance appears in the relativistic gravitational problem as well as the quantum electromagnetic theory forecasts splitting of the spectrum for a hydrogen atom. These facts indicate indirectly the absence of internal symmetries for such interactions. The scalar-vector interaction does not occur in nature purely but can contribute to some effective potentials such as quark-antiquark ones. Thus, the model proposed in the present work could be useful for description of hadrons, and internal symmetries may simplify the constructing of an approximation scheme on the quantum level.

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