LULEÅ UNIVERSITY OF TECHNOLOGY
Division of Physics

Solution to written exam in QUANTUM PHYSICS MTF067

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1. Hydrogenic atoms have eigenfunctions $\psi_{nlm} = R_{nl}(r)Y_{lm}(\theta,\varphi)$. Using the COLLECTION OF FORMULAE we find

$$\begin{aligned}
\psi_{100}(\boldsymbol{r}) &= \left(\tfrac{Z^3}{\pi a_0^3}\right)^{1/2} e^{-Zr/a_0} \\
\psi_{200}(\boldsymbol{r}) &= \left(\tfrac{Z^3}{8\pi a_0^3}\right)^{1/2} \left(1-\tfrac{Zr}{2a_0}\right) e^{-Zr/2a_0} \\
\psi_{210}(\boldsymbol{r}) &= \left(\tfrac{Z^3}{32\pi a_0^3}\right)^{1/2} \tfrac{Zr}{a_0} \cos\theta e^{-Zr/2a_0} \\
\psi_{21\pm1}(\boldsymbol{r}) &= \left(\tfrac{Z^3}{\pi a_0^3}\right)^{1/2} \tfrac{Zr}{8a_0} \sin\theta e^{\pm i\varphi} e^{-Zr/2a_0}
\end{aligned}$$

where a_0 is the Bohr radius. The β-decay instantaneously changes $Z=1 \to Z=2$. According to the expansion theorem, it is possible to express the wave function $u_i(\boldsymbol{r})$ before the decay as a linear combination of eigenfunctions $v_j(\boldsymbol{r})$ after the decay as

$$u_i(\boldsymbol{r}) = \sum_j a_j v_j(\boldsymbol{r})$$

where

$$a_j = \int v_j^*(\boldsymbol{r}) u_i(\boldsymbol{r}) d^3r.$$

The probability to find the electron in state j is given by $|a_j|^2$.

(a) Here $u_i = \psi_{100}(Z=1)$ and $v_j = \psi_{200}(Z=2)$. This gives

$$\begin{aligned}
a &= \left(\frac{1}{\pi a_0^3}\right)^{1/2} \left(\frac{2^3}{8\pi a_0^3}\right)^{1/2} \int_0^\infty e^{-r/a_0} \left(1-\frac{2r}{2a_0}\right) e^{-2r/2a_0} 4\pi r^2 dr \\
&= \frac{4}{a_0^3} \int_0^\infty e^{-2r/a_0} \left(r^2 - \frac{r^3}{a_0}\right) dr = \frac{4}{a_0^3} \left[2\left(\frac{a_0}{2}\right)^3 - \frac{6}{a_0}\left(\frac{a_0}{2}\right)^4\right] = -\frac{1}{2}.
\end{aligned}$$

Thus, the searched probability is $1/4 = 0.25$.
(The probability to find the electron in $\psi_{100}(Z=2)$ is $512/729 = 0.702$. Therefore, the electron is found with 95% probability in one of the states 1s or 2s.)

(b) For $u_i = \psi_{100}(Z=1)$ and $v_j = \psi_{210}(Z=2)$ the θ-integral is

$$\int_0^\pi \cos\theta \sin\theta d\theta = \frac{1}{2}\int_0^\pi \sin 2\theta d\theta = \left[-\frac{\cos 2\theta}{4}\right]_0^\pi = 0.$$

For $u_i = \psi_{100}(Z=1)$ and $v_j = \psi_{21\pm1}(Z=2)$ the φ-integral is

$$\int_0^{2\pi} e^{\pm i\varphi} d\varphi = 0.$$

Thus, the probability to find the electron in a 2p state is zero.

2. (a) Use the spherical coordinates

$$\begin{cases} x &= r\sin\theta\cos\varphi \\ x &= r\sin\theta\sin\varphi \\ x &= r\cos\theta \end{cases}$$

to expand the wave function $\psi(x,y,z)$ as a sum of functions $R(r)Y_{lm}(\theta,\phi)$. This gives

$$\begin{aligned}\psi &= Nr^2\sin\theta\cos\varphi\sin\theta\sin\varphi e^{-\alpha r} = Nr^2\sin^2\theta\sin\varphi\cos\varphi e^{-\alpha r} \\ &= \tfrac{1}{2}Nr^2\sin^2\theta\sin 2\varphi e^{-\alpha r} = \tfrac{1}{2}N\sin^2\theta\tfrac{1}{2i}\left(e^{2i\varphi}-e^{-2i\varphi}\right)r^2e^{-\alpha r} \\ &= \tfrac{1}{2}N\tfrac{1}{2i}\sqrt{\tfrac{32\pi}{15}}\left(Y_{22}-Y_{2-2}\right)r^2e^{-\alpha r} = N'\left(Y_{22}-Y_{2-2}\right)r^2e^{-\alpha r},\end{aligned}$$

where Y_{lm} are taken from the COLLECTION OF FORMULAS. From the expansion we see that the only possible values of l and m are $l=2$ and $m=\pm 2$.
Therefore, a measurement of L^2 will give the value $L^2 = 2(2+1)\hbar^2 = 6\hbar^2$ with probability $P(L^2 = 6\hbar^2) = 1$. A measurement of L_z will give the values $L_z = \pm 2\hbar$ with probabilities $P(L_z = +2\hbar) = P(L_z = -2\hbar) = 1/2$.

(b) The expectation value can be calculated as $\langle A\rangle = \sum_i P_iA_i$, where P_i is the probability to measure the value A_i. Therefore we have
$\langle L^2\rangle = 1\cdot 6\hbar^2 = 6\hbar^2$ and
$\langle L_z\rangle = 1/2\cdot 2\hbar + 1/2\cdot(-2\hbar) = 0$.

(c) The Schrödinger equation in spherical coordinates is given by

$$\left(-\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial^2}{\partial r^2}r\cdot + \frac{\boldsymbol{L}^2}{2mr^2} + V(r)\right)N'\left(Y_{22}-Y_{2-2}\right)r^2e^{-\alpha r} = EN'\left(Y_{22}-Y_{2-2}\right)r^2e^{-\alpha r}$$

If we use the fact that $L^2\psi = 6\hbar^2$ and that

$$-\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r^3e^{-\alpha r}\right) = -\frac{\hbar^2}{2m}\left(\frac{6}{r^2}-\frac{6\alpha}{r}+\alpha^2\right),$$

we can write the Schrödinger equation as

$$\left\{-\frac{\hbar^2}{2m}\left(\frac{6}{r^2}-\frac{6\alpha}{r}+\alpha^2\right)+\frac{6\hbar^2}{2mr^2}+V(r)-E\right\}\psi = 0,$$

or

$$-\frac{\hbar^2}{2m}\left(\frac{6}{r^2}-\frac{6\alpha}{r}+\alpha^2\right)+\frac{6\hbar^2}{2mr^2}+V(r)-E = 0. \qquad (1)$$

Since the spherically symmetric potential $V(r)\to 0$ as $r\to\infty$ we have

$$E = -\frac{\hbar^2\alpha^2}{2m}.$$

When this result is inserted into equation (1) we get

$$V(r) = -\frac{3\alpha\hbar^2}{mr}.$$

3. A measurement of the spin component in the direction $\hat{n} = \cos\varphi\hat{x} + \sin\varphi\hat{y}$ gives the value $\hbar/2$. The spin operator $S_{\hat{n}}$ is

$$S_{\hat{n}} = \frac{\hbar}{2}\begin{pmatrix} 0 & \cos\varphi - i\sin\varphi \\ \cos\varphi - i\sin\varphi & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & e^{-i\varphi} \\ e^{i\varphi} & 0 \end{pmatrix}$$

The eigenvalue equation is

$$S_{\hat{n}}\chi = \lambda\chi \Leftrightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & e^{-i\varphi} \\ e^{i\varphi} & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix} \tag{2}$$

We find the eigenvalues from

$$\begin{vmatrix} -\lambda & \frac{\hbar}{2}e^{-i\varphi} \\ \frac{\hbar}{2}e^{i\varphi} & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda = \pm\frac{\hbar}{2}$$

(a) The spin state corresponding to $\lambda = +\hbar/2$ must satisfy the eigenvalue equation Eq. (2), *i.e.*

$$\chi_{\hat{n}+} = \begin{pmatrix} a \\ b \end{pmatrix} = b\begin{pmatrix} e^{-i\varphi} \\ 1 \end{pmatrix} \Rightarrow \chi_{\hat{n}+} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\varphi} \\ 1 \end{pmatrix},$$

where the normalization condition $|a|^2 + |b|^2 = 1$ was used in the last step. Other correct solutions can be found by a multiplication with an arbitrary phase factor $\exp(i\alpha)$.

(b) A general spin state can be written as $\chi = a\chi_+ + b\chi_-$, where χ_+ is spin up and χ_- is spin down in z-direction. For $\chi_{\hat{n}+}$ we find that the probability to measure spin up, *i.e.* $S_z = \hbar/2$ is $|a|^2 = |e^{-i\varphi}/\sqrt{2}|^2 = 1/2$, and that the probability to measure spin down, *i.e.* $S_z = -\hbar/2$ is $|b|^2 = |1/\sqrt{2}|^2 = 1/2$.

4. (a) For a hydrogen atom we have the quantum numbers n, l, m, and s. Here we neglect the spin quantum number s. From FYSIKALIA we find that the energy is given by

$$E_n = -\frac{\mu e^4}{8\varepsilon_0^2 h^2} \cdot \frac{Z^2}{n^2} = -13.6 \cdot \frac{Z^2}{n^2}\ \text{eV}.$$

For a given n we have $l = 0, 1, 2, ..., n-1$ and for a given l we have $m = -l, -l+1, ..., l-1, l$. Thus, for a given n there is a n-fold l-degeneracy and each state with a given l exhibits a $(2l+1)$-fold m-degeneracy. Therefore, the energy level E_n is

$$\sum_{l=0}^{n-1}(2l+1) = n^2\text{-fold degenerate.}$$

(b) The first order energy correction $E_n^{(1)}$ for the perturbation H_1 is

$$E_n^{(1)} = \langle\phi_n|H_1|\phi_n\rangle\,,$$

if the Hamiltonian is $H = H_0 + H_1$ and ϕ_n are eigenstates to H_0. Here we have $H_1 = \epsilon/r^3$ and $\phi = \psi_{nlm} = R_{nl}(r)Y_{lm}(\theta,\varphi)$. Thus,

$$E_n^{(1)} = \epsilon\left\langle\psi_{nlm}|r^{-3}|\psi_{nlm}\right\rangle = \frac{\epsilon}{n^3 l(l+1/2)(l+1))}\left(\frac{Z}{a_0}\right)^3.$$

The expectation value $\langle r^{-3}\rangle$ is taken from the COLLECTION OF FORMULAE.

(c) The the spin-orbit interaction will cause a small amount of energy splitting between the l-degenerate states.

5. (a) For relativistic electrons the energy is given by $E = pc = \hbar ck$. The boundary conditions for a one-dimensional box of length L give $kL = n\pi$. For the three-dimensional box we have the same condition in each direction. Therefore, the energy will be given by

$$E = \frac{\hbar c\pi}{L}(n_1 + n_2 + n_3),$$

where n_1, n_2, and n_3 are positive integers. If $\bar{n} = (n_1, n_2, n_3)$ is a vector in a three-dimensional space, then each state will occupy a volume $V_{state} = 1$. The Fermi-energy is the border between filled and empty states. We can put two electrons in each state and the number of states with $|(n_1, n_2, n_3)| \leq R_F$ is

$$\frac{N}{2} = \frac{1}{8}\frac{4\pi}{3}R_F^3 \Rightarrow R_F = \left(\frac{3N}{\pi}\right)^{1/3}$$

Now the Fermi-energy can be calculated as

$$E_F = \frac{\hbar c\pi}{L}(n_1 + n_2 + n_3) = \frac{\hbar c\pi}{L}R_F = \frac{\hbar c\pi}{L}\left(\frac{3N}{\pi}\right)^{1/3} = \hbar c\left(\frac{3\pi^2 N}{V}\right)^{1/3}.$$

(b) The total energy of N relativistic electrons in the box is given by

$$\begin{aligned} E_{tot} &= 2\frac{1}{8}\int_{|\bar{n}|\leq R_F} E(\bar{n})d^3n = \frac{1}{4}\frac{\hbar c\pi}{L}\int_0^{R_F} n4\pi n^2 dn = \frac{\hbar c\pi^2}{L}\int_0^{R_F} n^3 dn \\ &= \frac{\hbar c\pi^2}{4L}R_F^4 = \frac{\hbar c\pi^2}{4L}\left(\frac{3N}{\pi}\right)^{4/3} = \frac{\hbar c\pi^2}{4}\left(\frac{3N}{\pi}\right)^{4/3}V^{-1/3} = \frac{V\hbar c}{4\pi^2}\left(\frac{3\pi^2 N}{V}\right)^{4/3}. \end{aligned}$$

(c) The degeneracy pressure is

$$p_{deg} = -\frac{\partial E_{tot}}{\partial V} = -\frac{\partial}{\partial V}\frac{\hbar c\pi^2}{4}\left(\frac{3N}{\pi}\right)^{4/3}V^{-1/3} = \frac{1}{3}\frac{\hbar c\pi^2}{4}\left(\frac{3N}{\pi}\right)^{4/3}V^{-4/3}$$

At *the Chandrasekhar limit* $p_{deg} + p_g = 0$, so

$$\frac{1}{3}\frac{\hbar c\pi^2}{4}\left(\frac{3N}{\pi}\right)^{4/3}V^{-4/3} = 0.69 \cdot \frac{1}{3}\left(\frac{4\pi}{3}\right)^{1/3}GM^2V^{-4/3}$$

Since we have one proton and one neutron per electron, the total number of electrons in the star is $N = M/2u$, where M is the mass of the star and $2u$ is the mass of one proton and one neutron. Thus

$$\begin{aligned} \frac{\hbar c\pi^2}{4}\left(\frac{3M}{2u\pi}\right)^{4/3} &= 0.69 \cdot \left(\frac{4\pi}{3}\right)^{1/3}GM^2 \Rightarrow \\ M^{2/3} &= \frac{\frac{\hbar c\pi^2}{4}\left(\frac{3}{2u\pi}\right)^{4/3}}{0.69 \cdot \left(\frac{4\pi}{3}\right)^{1/3}G} \Rightarrow \\ M^{2/3} &= \frac{\hbar c\pi^2}{4}\left(\frac{3}{2u\pi}\right)^{4/3}\left(\frac{3}{4\pi}\right)^{1/3}\frac{1}{0.69 \cdot G} \Rightarrow \\ M^{2/3} &= \frac{\hbar c}{3\pi}\left(\frac{9\pi}{8u}\right)^{4/3}\frac{1}{0.69 \cdot G} \Rightarrow \end{aligned}$$

$$M = 2.82 \cdot 10^{30}\ \text{kg} = 1.4M_\odot,$$

where $M_\odot = 1.989 \cdot 10^{30}$ kg.