LULEÅ UNIVERSITY OF TECHNOLOGY
Division of Physics

Solution to written exam in QUANTUM PHYSICS MTF067

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1. The energy eigenvalues for a particle in a box are given by

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2 n^2}{2ma^2}, \text{ where } n = 1, 2, 3, ...$$

   (a) There is no limit on the number of bosons in one state. Therefore, the ground state energy of five bosons is $5E_1 = \frac{5\hbar^2\pi^2}{2ma^2}$.

   (b) Fermions obey the Pauli principle. Thus, the ground state energy of five fermions is given by

$$E = 2E_1 + 2E_2 + E_3 = 2\frac{\hbar^2\pi^2 1^2}{2ma^2} + 2\frac{\hbar^2\pi^2 2^2}{2ma^2} + 1\frac{\hbar^2\pi^2 3^2}{2ma^2} = \frac{19\hbar^2\pi^2}{2ma^2}.$$

2. Use a test function $f(r, \theta, \phi)$ to calculate the commutator.

$$\begin{aligned}
[L_z, \sin\phi] f(r,\theta,\phi) &= \left[-i\hbar\frac{\partial}{\partial\phi}, \sin\phi\right] f(r,\theta,\phi) \\
&= -i\hbar\frac{\partial}{\partial\phi}\sin\phi f(r,\theta,\phi) - \sin\phi\left(-i\hbar\frac{\partial}{\partial\phi}\right) f(r,\theta,\phi) \\
&= -i\hbar\cos\phi f - i\hbar\sin\phi\frac{\partial f}{\partial\phi} + i\hbar\sin\phi\frac{\partial f}{\partial\phi} \\
&= -i\hbar\cos\phi f(r,\theta,\phi).
\end{aligned}$$

   Thus, $[L_z, \sin\phi] = -i\hbar\cos\phi$.

3. The eigenfunctions and eigenvalues of the free-particle Hamiltonian are found by solving the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2u(x)}{dx^2} + V(x)u(x) = Eu(x),$$

   with $V(x)$ zero everywhere. Thus, the eigenvalue equation reads

$$\frac{d^2u(x)}{dx^2} + k^2u(x) = 0,$$

   where $k^2 = 2mE/\hbar^2$. The eigenfunctions are given by the plane waves e^{ikx} and e^{-ikx}, or linerar combinatoins of these, as *e.g.* $\sin kx$ and $\cos kx$.

   (a) The wave function of the particle at $t = 0$ is given by

$$\psi(x, 0) = \cos^3 kx$$

   which can be written as

$$\psi(x,0) = \left(\frac{e^{ikx} + e^{-ikx}}{2}\right)^3 = \frac{1}{8}\left(e^{i3kx} + 3e^{ikx} + 3e^{-ikx} + e^{-i3kx}\right) = \frac{3}{4}\cos kx + \frac{1}{4}\cos 3kx. \quad (1)$$

   Thus, $\psi(x, 0)$ can be written as a superposition the plane waves $\cos k_1x$ and $\cos k_2x$, with $k_1 = k$ and $k_2 = 3k$

(b) The energy of a plane wave e^{ikx} is given by $E = \hbar^2 k^2/2m$. Thus, the energy of $\cos k_1 x$ is $E_1 = \hbar^2 k^2/2m$ and the energy of $\cos k_2 x$ is $E_2 = \hbar^2 k_2^2/2m = 9\hbar^2 k^2/2m$.

(c) $u(x) = e^{ikx}$ is a solution to the the time-independent Schrödinger equation. The corresponding solutions to the time-dependent Schrödinger equation is given by $u(x)T(t)$,with $T(t) = e^{-iEt/\hbar}$. Therefore, $u(x)T(t) = e^{i(kx-Et/\hbar)}$. A sum of solutions of this form is also a solution, since the Schrödinger equation is linear. This means that if $\psi(x,0)$ is given by equation (1), then

$$\begin{aligned}\psi(x,t) &= \frac{1}{8}\left[\left(e^{i3kx}+e^{-i3kx}\right)e^{-iE_2t/\hbar}+3\left(e^{ikx}+e^{-ikx}\right)e^{-iE_1t/\hbar}\right] \\ &= \frac{3}{4}\cos(kx)e^{-iE_1t/\hbar}+\frac{1}{4}\cos(3kx)e^{-iE_2t/\hbar} \qquad (2)\end{aligned}$$

where

$$E_1 = \frac{\hbar^2 k^2}{2m} \quad \text{and} \quad E_2 = \frac{9\hbar^2 k^2}{2m} \qquad (3)$$

4. (a) The solution to the time-independent Schrödinger equation is

$$\begin{cases} u(x) = e^{ikx} + Re^{-ikx} & ,k = \frac{\sqrt{2mE}}{\hbar} & ,x<0, \\ u(x) = Te^{iqx} & ,q = \frac{\sqrt{2m(E-V_0)}}{\hbar} & ,x>0 \end{cases}$$

The continuity conditions at $x = 0$ give

$$\begin{cases} T = 1 + R \\ iqT = ik(1-R) \end{cases} \Rightarrow T = \frac{2k}{k+q}\ , \text{ and } R = \frac{k-q}{k+q}$$

Thus,

$$\begin{cases} u(x) = e^{ikx} + \frac{k-q}{k+q}e^{-ikx} & ,x<0, \\ u(x) = \frac{2k}{k+q}e^{iqx} & ,x>0 \end{cases}$$

(b) The transmitted part is given by the transmitted probability flux divided by the incoming probability flux, i.e.

$$\frac{\hbar q}{m}|T|^2 \Big/ \frac{\hbar k}{m} = \frac{q}{k}|T|^2 = \frac{4qk}{(k+q)^2} = \frac{4qk}{(k+q)^2}.$$

Therefore, the number of transmitted particles per time unit is

$$\frac{4qk}{(k+q)^2}N_0.$$

5. A measurement of the spin component in the direction $\hat{n} = \cos\varphi\hat{x} + \sin\varphi\hat{y}$ gives the value $\hbar/2$. The spin operator $S_{\hat{n}}$ is

$$S_{\hat{n}} = \frac{\hbar}{2}\begin{pmatrix} 0 & \cos\varphi - i\sin\varphi \\ \cos\varphi - i\sin\varphi & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & e^{-i\varphi} \\ e^{i\varphi} & 0 \end{pmatrix}$$

The eigenvalue equation is

$$S_{\hat{n}}\chi = \lambda\chi \Leftrightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & e^{-i\varphi} \\ e^{i\varphi} & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix} \tag{4}$$

We find the eigenvalues from

$$\begin{vmatrix} -\lambda & \frac{\hbar}{2}e^{-i\varphi} \\ \frac{\hbar}{2}e^{i\varphi} & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda = \pm\frac{\hbar}{2}$$

(a) The spin state corresponding to $\lambda = +\hbar/2$ must satisfy the eigenvalue equation Eq. (4), *i.e.*

$$\chi_{\hat{n}+} = \begin{pmatrix} a \\ b \end{pmatrix} = b\begin{pmatrix} e^{-i\varphi} \\ 1 \end{pmatrix} \Rightarrow \chi_{\hat{n}+} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\varphi} \\ 1 \end{pmatrix},$$

where the normalization condition $|a|^2 + |b|^2 = 1$ was used in the last step. Other correct solutions can be found by a multiplication with an arbitrary phase factor $\exp(i\alpha)$.

(b) A general spin state can be written as $\chi = a\chi_+ + b\chi_-$, where χ_+ is spin up and χ_- is spin down in z-direction. For $\chi_{\hat{n}+}$ we find that the probability to measure spin up, *i.e.* $S_z = \hbar/2$ is $|a|^2 = |e^{-i\varphi}/\sqrt{2}|^2 = 1/2$, and that the probability to measure spin down, *i.e.* $S_z = -\hbar/2$ is $|b|^2 = |1/\sqrt{2}|^2 = 1/2$.

6. Hydrogenic atoms have eigenfunctions $\psi_{nlm} = R_{nl}(r)Y_{lm}(\theta,\varphi)$. Using the COLLECTION OF FORMULAE we find

$$\begin{aligned}
\psi_{100}(\boldsymbol{r}) &= \left(\tfrac{Z^3}{\pi a_0^3}\right)^{1/2} e^{-Zr/a_0} \\
\psi_{200}(\boldsymbol{r}) &= \left(\tfrac{Z^3}{8\pi a_0^3}\right)^{1/2} \left(1 - \tfrac{Zr}{2a_0}\right) e^{-Zr/2a_0} \\
\psi_{210}(\boldsymbol{r}) &= \left(\tfrac{Z^3}{32\pi a_0^3}\right)^{1/2} \tfrac{Zr}{a_0} \cos\theta e^{-Zr/2a_0} \\
\psi_{21\pm1}(\boldsymbol{r}) &= \left(\tfrac{Z^3}{\pi a_0^3}\right)^{1/2} \tfrac{Zr}{8a_0} \sin\theta e^{\pm i\varphi} e^{-Zr/2a_0}
\end{aligned}$$

where a_0 is the Bohr radius. The β-decay instantaneously changes $Z = 1 \rightarrow Z = 2$. According to the expansion theorem, it is possible to express the wave function $u_i(\boldsymbol{r})$ before the decay as a linear combination of eigenfunctions $v_j(\boldsymbol{r})$ after the decay as

$$u_i(\boldsymbol{r}) = \sum_j a_j v_j(\boldsymbol{r})$$

where

$$a_j = \int v_j^*(\boldsymbol{r}) u_i(\boldsymbol{r}) d^3r.$$

The probability to find the electron in state j is given by $|a_j|^2$.

(a) Here $u_i = \psi_{100}(Z = 1)$ and $v_j = \psi_{200}(Z = 2)$. This gives

$$\begin{aligned}
a &= \left(\frac{1}{\pi a_0^3}\right)^{1/2} \left(\frac{2^3}{8\pi a_0^3}\right)^{1/2} \int_0^\infty e^{-r/a_0} \left(1 - \frac{2r}{2a_0}\right) e^{-2r/2a_0} 4\pi r^2 dr \\
&= \frac{4}{a_0^3} \int_0^\infty e^{-2r/a_0} \left(r^2 - \frac{r^3}{a_0}\right) dr = \frac{4}{a_0^3}\left[2\left(\frac{a_0}{2}\right)^3 - \frac{6}{a_0}\left(\frac{a_0}{2}\right)^4\right] = -\frac{1}{2}.
\end{aligned}$$

Thus, the searched probability is $1/4 = 0.25$.
(The probability to find the electron in $\psi_{100}(Z=2)$ is $512/729 = 0.702$. Therefore, the electron is found with 95% probability in one of the states 1s or 2s.)

(b) For $u_i = \psi_{100}(Z=1)$ and $v_j = \psi_{210}(Z=2)$ the θ-integral is

$$\int_0^\pi \cos\theta \sin\theta d\theta = \frac{1}{2}\int_0^\pi \sin 2\theta d\theta = \left[-\frac{\cos 2\theta}{4}\right]_0^\pi = 0.$$

For $u_i = \psi_{100}(Z=1)$ and $v_j = \psi_{21\pm1}(Z=2)$ the φ-integral is

$$\int_0^{2\pi} e^{\pm i\varphi} d\varphi = 0.$$

Thus, the probability to find the electron in a 2p state is zero.