LULEÅ UNIVERSITY OF TECHNOLOGY
Division of Physics

Course code	MTF067
Examination date	2001-12-17
Time	09.00 - 14.00

Solutions to Quantum Physics

1. The spherical harmonics can generally be written

$$Y_{lm} = N_{lm}\, f(\theta)\, e^{im\phi},$$

where $f(\theta)$ is a function of θ *only*, and N_{lm} is the normalization constant.
From *COLLECTION OF FORMULAE*

$$L_z = -i\hbar \frac{\partial}{\partial \phi},$$

so

$$L_z Y_{lm} = m\hbar Y_{lm}.$$

This means that the eigenvalue of L_z is $m\hbar$, and the interpretation is that this is the z-component of the angular momentum of the quantum mechanical system.

2. We are to calculate

$$\langle x \rangle = \langle u_n | x | u_n \rangle.$$

and

$$\langle x^2 \rangle = \langle u_n | x^2 | u_n \rangle.$$

From *COLLECTION OF FORMULAE*

$$x|u_n\rangle = \frac{b}{\sqrt{2}}(\sqrt{n+1}|u_{n+1}\rangle + \sqrt{n}|u_{n-1}\rangle).$$

But, because of the orthogonality of the eigenfunctions

$$\langle u_m | u_n \rangle = \delta_{mn},$$

this gives

$$\langle u_n | x | u_n \rangle = 0.$$

From *COLLECTION OF FORMULAE* we also get

$$x^2|u_n\rangle = \frac{b^2}{2}(\sqrt{(n+1)(n+2)}|u_{n+2}\rangle + (2n+1)|u_n\rangle + \sqrt{n(n-1)}|u_{n-2}\rangle).$$

The only term that "survives" when we take the scalar product with $\langle u_n|$ is $(2n+1)|u_n\rangle$, so

$$\langle x^2 \rangle = \langle u_n | x^2 | u_n \rangle = \frac{b^2}{2}(2n+1) = \frac{\hbar}{2m\omega}(2n+1) = \frac{\hbar}{m\omega}(n + \frac{1}{2}).$$

3. As the wave function is given, we can calculate the expectation values directly as the (probability) weighted sums of the different eigenvalues. Eigenfunctions: ψ_{nlm}, where n gives the energy, l gives the total angular momentum squared according to $l(l+1)\hbar^2$, and m gives the z-component of the angular momentum, $m\hbar$.

   a) Expectation value for the energy

$$\langle E \rangle = (\frac{4}{6})^2 E_1 + (\frac{3}{6})^2 E_2 + (\frac{-1}{6})^2 E_2 + (\frac{\sqrt{10}}{6})^2 E_2 = \frac{4}{9} E_1 + \frac{5}{9} E_2.$$

   For the hydrogen atom, the energies are $E_n \approx -13.6/n^2 \ eV$, so

$$\langle E \rangle \approx -7.9 \, eV.$$

   b) Expectation value for the total angular momentum squared

$$\langle \mathbf{L}^2 \rangle = \frac{4}{9} \cdot 0 + \frac{5}{9} \cdot 1(1+1)\hbar^2 = \frac{10}{9} \hbar^2.$$

   c) Expectation value for the z-component of the angular momentum

$$\langle L_z \rangle = (\frac{4}{9} + \frac{1}{36}) \cdot 0 + (\frac{3}{6})^2 \cdot 1 \cdot \hbar + (\frac{\sqrt{10}}{6})^2 \cdot (-1) \cdot \hbar = -\frac{1}{36}\hbar.$$

4. a) Use *Pauli matrices*, see textbook.

   b) Spin eigenvalues in general: $s(s+1)\hbar^2$ for $\mathbf{S}^2$; $m_s\hbar$ for S_z. For spin-1/2, s=1/2, which gives $\frac{3}{4}\hbar^2$ for $\mathbf{S}^2$, and $\pm\hbar/2$ for S_z. As the operators commute, they are simultaneously measurable.

   c) The probability for $\mathbf{S}^2$ being $\frac{3}{4}\hbar^2$ is one. The probability for S_z being $+\hbar/2$ is $(\frac{2}{\sqrt{5}})^2 = \frac{4}{5}$. The probability for S_z being $-\hbar/2$ is $|\frac{i}{\sqrt{5}}|^2 = \frac{1}{5}$.

5. See, *e.g.*, solution to problem 6, exam 2001-04-18.