LULEÅ UNIVERSITY OF TECHNOLOGY
Division of Physics

Course code	MTF067
Examination date	2002-04-24
Time	09.00 - 14.00

Solutions to Quantum Physics

1. A time-independent potential, separation of variables $\psi(x,t) = u(x)\,T(t)$ and the time-dependent Schrödinger equation $[i\hbar\frac{\partial}{\partial t}\psi(x,t) = H\psi(x,t)]$ gives: $Hu(x) = Eu(x)$, the stationary Schrödinger equation.

   (See, e.g., Gasiorowicz pp. 54-55, for mathematical details.)

2. We are to calculate

$$\langle x\rangle = \langle u_n|x|u_n\rangle.$$

   and

$$\langle x^2\rangle = \langle u_n|x^2|u_n\rangle.$$

   From *COLLECTION OF FORMULAE*

$$x|u_n\rangle = \frac{b}{\sqrt{2}}(\sqrt{n+1}|u_{n+1}\rangle + \sqrt{n}|u_{n-1}\rangle).$$

   But, because of the orthogonality of the eigenfunctions

$$\langle u_m|u_n\rangle = \delta_{mn},$$

   this gives

$$\langle u_n|x|u_n\rangle = 0.$$

   From *COLLECTION OF FORMULAE* we also get

$$x^2|u_n\rangle = \frac{b^2}{2}(\sqrt{(n+1)(n+2)}|u_{n+2}\rangle + (2n+1)|u_n\rangle + \sqrt{n(n-1)}|u_{n-2}\rangle).$$

   The only term that "survives" when we take the scalar product with $\langle u_n|$ is $(2n+1)|u_n\rangle$, so

$$\langle x^2\rangle = \langle u_n|x^2|u_n\rangle = \frac{b^2}{2}(2n+1) = \frac{\hbar}{2m\omega}(2n+1) = \frac{\hbar}{m\omega}(n+\frac{1}{2}).$$

   a) For the ground state, $n = 0$, giving $\langle x\rangle = 0$, $\langle x^2\rangle = \frac{\hbar}{2m\omega}$.

   b) For the first excited state, $n = 1$, so $\langle x\rangle = 0$, $\langle x^2\rangle = \frac{3\hbar}{2m\omega}$.

3. As the wave function is given, we can calculate the expectation values directly as the (probability) weighted sums of the different eigenvalues. Eigenfunctions: ψ_{nlm}, where n gives the energy, l gives the total angular momentum squared according to $l(l+1)\hbar^2$, and m gives the z-component of the angular momentum, $m\hbar$.

   a) Expectation value for the energy

$$\langle E\rangle = (\frac{4}{6})^2E_1 + (\frac{3}{6})^2E_2 + (\frac{-1}{6})^2E_2 + (\frac{\sqrt{10}}{6})^2E_2 = \frac{4}{9}E_1 + \frac{5}{9}E_2.$$

For the hydrogen atom, the energies are $E_n \approx -13.6/n^2\ eV$, so

$$\langle E \rangle \approx -7.9\, eV.$$

b) Expectation value for the total angular momentum squared

$$\langle \mathbf{L}^2 \rangle = \frac{4}{9} \cdot 0 + \frac{5}{9} \cdot 1(1+1)\hbar^2 = \frac{10}{9}\,\hbar^2.$$

c) Expectation value for the z-component of the angular momentum

$$\langle L_z \rangle = (\frac{4}{9} + \frac{1}{36}) \cdot 0 + (\frac{3}{6})^2 \cdot 1 \cdot \hbar + (\frac{\sqrt{10}}{6})^2 \cdot (-1) \cdot \hbar = -\frac{1}{36}\hbar.$$

4. a) $1 = |\chi|^2 = N^2(2^2 + |i|^2) = 5N^2$, giving $N = \frac{1}{\sqrt{5}}$.

   b) Spin eigenvalues in general: $s(s+1)\hbar^2$ for $\mathbf{S}^2$; $m_s\hbar$ for S_z. For spin-1/2, s=1/2, which gives $\frac{3}{4}\hbar^2$ for $\mathbf{S}^2$, and $\pm\hbar/2$ for S_z. As $[\mathbf{S}^2, S_z] = 0$, they are simultaneously measurable.

   c) The probability for $\mathbf{S}^2$ being $\frac{3}{4}\hbar^2$ is one. The probability for S_z being $+\hbar/2$ is $(\frac{2}{\sqrt{5}})^2 = \frac{4}{5}$. The probability for S_z being $-\hbar/2$ is $|\frac{i}{\sqrt{5}}|^2 = \frac{1}{5}$.

5. Rewrite $L_x^2 + L_y^2 = \mathbf{L}^2 - L_z^2$, which gives

$$H = \frac{\mathbf{L}^2 - L_z^2}{2I_1} + \frac{L_z^2}{2I_2},$$

so

$$HY_{l,m} = (\frac{l(l+1)\hbar^2 - m^2\hbar^2}{2I_1} + \frac{m^2\hbar^2}{2I_2})Y_{l,m},$$

giving

$$E_{l,m} = \hbar^2[\frac{l(l+1)}{2I_1} + \frac{m^2}{2}(\frac{1}{I_2} - \frac{1}{I_1})].$$

The lowest energy ($l = m = 0$) is *zero* (no rotation)!

$l = 1 \Rightarrow m = 0, \pm 1$

$l = 2 \Rightarrow m = 0, \pm 1, \pm 2$

and so on.