LULEÅ UNIVERSITY OF TECHNOLOGY
Division of Physics

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Solutions to Quantum Physics

1. Two observables, A,B, are simultaneously measurable only if their corresponding operators commute, *i.e.* if $[A, B] = 0$.

   a) $[\mathbf{S}^2, S_x] = [\mathbf{S}^2, S_y] = [\mathbf{S}^2, S_z] = 0$. (As $\mathbf{S}^2 = S_x^2 + S_y^2 + S_z^2$ is the unit 2×2 matrix multiplied by $\frac{3}{4}\hbar^2$.)

   b) $[S_x, S_y] = i\hbar S_z$, $[S_x, S_z] = -i\hbar S_y$, $[S_y, S_z] = i\hbar S_x$. (That is, if you choose to measure for instance S_x you cannot simultaneously measure S_y, S_z.)

2. a) The eigenvalues of S_z are $+\hbar/2$, $-\hbar/2$.

   b) The normalization constant $N = 1/\sqrt{10}$. The probability for $+\hbar/2$ is $P(+\hbar/2) = (3/\sqrt{10})^2 = 9/10$. The probability for $-\hbar/2$ is $P(-\hbar/2) = (1/\sqrt{10})^2 = 1/10$.

   c) The expectation value $\langle S_z \rangle = \frac{9}{10}\hbar/2 + \frac{1}{10}(-\hbar/2) = \frac{4}{5}\hbar/2 = \frac{2}{5}\hbar$.

3. a) $\langle V \rangle = \frac{1}{2}k\langle x^2 \rangle = \frac{1}{2}m\omega^2\langle x^2 \rangle$.

   $\langle x^2 \rangle = \langle u_n|x^2|u_n \rangle =$ [using *Collection of formulae*] $= \langle u_n|\frac{b^2}{2}(2n+1)|u_n\rangle =$ [where $b^2 = \frac{\hbar}{m\omega}$, with the last step resulting because $\langle u_n|u_m\rangle = 0$ if $m \neq n$] $= \frac{b^2}{2}(2n + 1)\langle u_n|u_n\rangle = \frac{b^2}{2}(2n+1)$.

   So, $\langle V \rangle = \frac{1}{2}m\omega^2\frac{\hbar}{2m\omega}(2n+1) = \frac{\hbar\omega}{2}(n+\frac{1}{2})$.

   b) $\langle E \rangle = \langle K \rangle + \langle V \rangle$, which gives $\langle K \rangle = \langle E \rangle - \langle V \rangle = (n+\frac{1}{2})\hbar\omega - (n+\frac{1}{2})\hbar\omega/2 = (n+\frac{1}{2})\hbar\omega/2 = \langle V \rangle$.

4. As the wave function is given, we can calculate the expectation values directly as the (probability) weighted sums of the different eigenvalues. Eigenfunctions: ψ_{nlm}, where n gives the energy, l gives the total angular momentum squared according to $l(l+1)\hbar^2$, and m gives the z-component of the angular momentum, $m\hbar$.

   a) Expectation value for the energy

$$\langle E\rangle = (\frac{4}{6})^2 E_1 + (\frac{3}{6})^2 E_2 + (\frac{-1}{6})^2 E_2 + (\frac{\sqrt{10}}{6})^2 E_2 = \frac{4}{9}E_1 + \frac{5}{9}E_2.$$

   For the hydrogen atom, the energies are $E_n \approx -13.6/n^2\ eV$, so

$$\langle E\rangle \approx -7.9\, eV.$$

   b) Expectation value for the total angular momentum squared

$$\langle \mathbf{L}^2\rangle = \frac{4}{9}\cdot 0 + \frac{5}{9}\cdot 1(1+1)\hbar^2 = \frac{10}{9}\,\hbar^2.$$

   c) Expectation value for the z-component of the angular momentum

$$\langle L_z\rangle = (\frac{4}{9} + \frac{1}{36})\cdot 0 + (\frac{3}{6})^2\cdot 1\cdot\hbar + (\frac{\sqrt{10}}{6})^2\cdot(-1)\cdot\hbar = -\frac{1}{36}\hbar.$$

5. Rewrite the wave function in terms of spherical harmonics

   $\psi(\mathbf{r}) = Cr^2e^{-\alpha r^2}(xy + yz + zx) = Cr^2e^{-\alpha r^2}[\frac{1}{4i}\sqrt{\frac{32\pi}{15}}Y_{2,2} - \frac{1}{4i}\sqrt{\frac{32\pi}{15}}Y_{2,-2} + \frac{1-i}{2}\sqrt{\frac{8\pi}{15}}Y_{2,1} + \frac{1+i}{2}\sqrt{\frac{8\pi}{15}}Y_{2,-1}]$

   a) None of the $Y_{l,m}$ have $l = 0$, so the probability for $l = 0$ is *zero*.

   b) $l = 2$ corresponds to $\mathbf{L}^2 = 2(2+1)\hbar^2 = 6\hbar^2$. As all $Y_{l,m}$ have $l = 2$ the probability for this is *one*.

   c) The relative probabilities (P) for different values of m are given by the ratios of the absolute squares of the corresponding coefficients. We see that $P(m = 0) = 0$ as the coefficient for $Y_{2,0}$ is zero. Also, $P(m = 2) = P(m = -2)$ and $P(m = 1) = P(m = -1)$.

   $P(m = 2)/P(m = 1) = (\frac{1}{16}\frac{32\pi}{15})/(\frac{1}{2}\frac{8\pi}{15}) = 1/2.$

   The probabilities must sum up to one

   $1 = P(m = 2) + P(m = -2) + P(m = 1) + P(m = -1) = 6P(m = 2)$, so

   $P(m = 2) = 1/6$, $P(m = -2) = 1/6$, $P(m = 1) = 1/3$, $P(m = -1) = 1/3$, $P(m = 0) = 0$.