

Course code	MTF067
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SOLUTIONS TO QUANTUM PHYSICS

1. a) The energies of a *bound* quantum system are *quantized*!
b) The Schrödinger equation is a *linear* differential equation, so the superposition principle is satisfied. (For example, if ψ_1 and ψ_2 are two different solutions to the Schrödinger equation, then $\psi_3 = \psi_1 + \psi_2$ is also a solution. This generalizes to arbitrarily many different solutions.)
c) More ways for something to happen can generate *destructive interference*, thus giving a lower probability.

2. As the wave function is given, we can calculate the expectation values directly as the (probability) weighted sums of the different eigenvalues. Eigenfunctions: ψ_{nlm} , where n gives the energy, l gives the total angular momentum squared according to $l(l+1)\hbar^2$, and m gives the z -component of the angular momentum, $m\hbar$.

- a) Expectation value for the energy

$$\langle E \rangle = \left(\frac{4}{6}\right)^2 E_1 + \left(\frac{3}{6}\right)^2 E_2 + \left(\frac{-1}{6}\right)^2 E_3 + \left(\frac{\sqrt{10}}{6}\right)^2 E_4 = \frac{4}{9} E_1 + \frac{1}{4} E_2 + \frac{1}{36} E_3 + \frac{5}{18} E_4.$$

For the hydrogen atom, the energies are $E_n \approx -13.6/n^2$ eV, so

$$\langle E \rangle \approx \frac{4}{9}(-13.6) + \frac{1}{4}(-3.4) + \frac{1}{36}(-1.5) + \frac{5}{18}(-0.8) \approx -7.2 \text{ eV}.$$

- b) Expectation value for the total angular momentum squared

$$\langle \mathbf{L}^2 \rangle = \frac{4}{9} \cdot 0 + \left(\frac{9}{36} + \frac{1}{36} + \frac{10}{36}\right) \cdot 1(1+1)\hbar^2 = \frac{10}{9} \hbar^2.$$

- c) Expectation value for the z -component of the angular momentum

$$\langle L_z \rangle = \left(\frac{16}{36} + \frac{1}{36} + \frac{10}{36}\right) \cdot 0 + \frac{9}{36} \cdot 1 \cdot \hbar = \frac{1}{4} \hbar.$$

3. a) The probability density is

$$\begin{aligned} \Psi^* \Psi &= [c_1^* \psi_1^*(x) e^{iE_1 t/\hbar} + c_2^* \psi_2^*(x) e^{iE_2 t/\hbar}] [c_1 \psi_1(x) e^{-iE_1 t/\hbar} + c_2 \psi_2(x) e^{-iE_2 t/\hbar}] = \\ &= c_1^* c_1 \psi_1^*(x) \psi_1(x) + c_2^* c_2 \psi_2^*(x) \psi_2(x) + c_2^* c_1 \psi_2^*(x) \psi_1(x) e^{i(E_2 - E_1)t/\hbar} + c_1^* c_2 \psi_1^*(x) \psi_2(x) e^{-i(E_2 - E_1)t/\hbar} = \\ &= |c_1|^2 |\psi_1(x)|^2 + |c_2|^2 |\psi_2(x)|^2 + 2 \operatorname{Re}(c_2^* c_1 \psi_2^*(x) \psi_1(x) e^{i(E_2 - E_1)t/\hbar}), \end{aligned}$$

where the last term shows that it oscillates in time.

b) $\omega = \frac{E_2 - E_1}{\hbar}$, so the oscillation frequency is

$$\nu = \frac{E_2 - E_1}{2\pi\hbar} = \frac{E_2 - E_1}{h}.$$

The energy of an emitted photon is $E = h\nu = E_2 - E_1$, which is the difference between the energy of the excited state and the ground state, as it should be.

4. a) $\langle V \rangle = \frac{1}{2} k \langle x^2 \rangle = \frac{1}{2} m \omega^2 \langle x^2 \rangle$.

$\langle x^2 \rangle = \langle u_n | x^2 | u_n \rangle = [\text{using } \textit{Collection of formulae}] = \langle u_n | \frac{b^2}{2} (2n + 1) | u_n \rangle = [\text{where } b^2 = \frac{\hbar}{m\omega}, \text{ with the last step resulting because } \langle u_n | u_m \rangle = 0 \text{ if } m \neq n] = \frac{b^2}{2} (2n + 1) \langle u_n | u_n \rangle = \frac{b^2}{2} (2n + 1)$.

So, $\langle V \rangle = \frac{1}{2} m \omega^2 \frac{\hbar}{2m\omega} (2n + 1) = \frac{\hbar\omega}{2} (n + \frac{1}{2})$.

b) $\langle E \rangle = \langle K \rangle + \langle V \rangle$, which gives $\langle K \rangle = \langle E \rangle - \langle V \rangle = (n + \frac{1}{2})\hbar\omega - (n + \frac{1}{2})\hbar\omega/2 = (n + \frac{1}{2})\hbar\omega/2 = \langle V \rangle$.

5. a) The radial probability density for the ground state ($n = 1, l = 0$) of Hydrogen ($Z = 1$) is:

$$P_{1,0}(r) = R_{1,0}^* R_{1,0}(r) 4\pi r^2$$

As we are only interested in the position of the maximum, we can ignore multiplicative (normalization) constants, and

$$P_{1,0}(r) = e^{-r/a_0} e^{-r/a_0} r^2 = e^{-2r/a_0} r^2$$

To evaluate the maximum we set the r -derivative equal to zero

$$0 = \frac{dP_{1,0}(r)}{dr} = (1 - \frac{r}{a_0}) e^{-2r/a_0} 2r$$

The maximum is given by $(1 - \frac{r}{a_0}) = 0$, or

$$r = a_0.$$

The maximum in the radial probability density for Hydrogen in its ground state lies at the “Bohr radius”, a_0 .

b) The expectation value of r for hydrogenic atoms is generally given by

$$\langle r \rangle = \frac{1}{2}[3n^2 - l(l+1)]\frac{a_0}{Z}$$

In our case, $n = 1$, $l = 0$, $Z = 1$, so

$$\langle r \rangle = \frac{3}{2}a_0$$

for Hydrogen in its ground state. The reason why the answer is not identical to that in a) is because the radial probability density is asymmetrical around a_0 , giving more weight to larger distances.