LULEÅ UNIVERSITY OF TECHNOLOGY
Division of Physics

Course code	MTF067
Examination date	2002-12-20
Time	09.00 - 14.00

Solutions to Quantum Physics

1. a) The general energy eigenvalue equation is the time-independent Schrödinger equation:

$$H\psi = E\psi,$$

where E are the energy eigenvalues (= the allowed energies of the system) and ψ the energy eigenstates.

b) The operator must be *Hermitian* (to assure that the eigenvalues are real).

c) One example is the step operator L_+, which gives another state than the one it acted on, making an eigenvalue equation (in simultaneous eigenstates of $\mathbf{L}^2$ and L_z) impossible.

2. As the wave function is given, we can calculate the expectation values directly as the (probability) weighted sums of the different eigenvalues. Eigenfunctions: ψ_{nlm}, where n gives the energy, l gives the total angular momentum squared according to $l(l+1)\hbar^2$, and m gives the z-component of the angular momentum, $m\hbar$.

We begin by calculating the normalization constant, N, by the requirement

$$\psi^*\psi = 1,$$

giving

$$|N|^2(4^2 + |3i|^2 + |-i|^2 + (\sqrt{10})^2) = 1,$$

or

$$N = \frac{1}{6}$$

a) Expectation value for the energy

$$\langle E \rangle = (\frac{4}{6})^2 E_1 + (\frac{|3i|}{6})^2 E_2 + (\frac{|-i|}{6})^2 E_3 + (\frac{\sqrt{10}}{6})^2 E_4 = \frac{4}{9}E_1 + \frac{1}{4}E_2 + \frac{1}{36}E_3 + \frac{5}{18}E_4.$$

For the hydrogen atom, the energies are $E_n \approx -13.6/n^2\ eV$, so

$$\langle E \rangle \approx \frac{4}{9}(-13.6) + \frac{1}{4}(-3.4) + \frac{1}{36}(-1.5) + \frac{5}{18}(-0.8) \approx -7.2\,eV.$$

b) Expectation value for the total angular momentum squared

$$\langle \mathbf{L}^2 \rangle = \frac{4}{9} \cdot 0 + (\frac{9}{36} + \frac{1}{36} + \frac{10}{36}) \cdot 1(1+1)\hbar^2 = \frac{10}{9}\hbar^2.$$

c) Expectation value for the z-component of the angular momentum

$$\langle L_z \rangle = (\frac{16}{36} + \frac{1}{36} + \frac{10}{36}) \cdot 0 + \frac{9}{36} \cdot 1 \cdot \hbar = \frac{1}{4}\hbar.$$

3. The spin-wavefunction as given is not normalized.

$$\chi^*\chi = 2^2 + |i|^2 = 5$$

Thus, a properly normalized state is

$$\chi = \frac{1}{\sqrt{5}}(2|\uparrow\rangle + i|\downarrow\rangle)$$

a) Use *Pauli matrices*, see textbook. Spin eigenvalues in general: $s(s+1)\hbar^2$ for $\mathbf{S}^2$; $m_s\hbar$ for S_z. For spin-1/2, s=1/2, which gives $\frac{3}{4}\hbar^2$ for $\mathbf{S}^2$, and $\pm\hbar/2$ for S_z. As the operators commute, $[\mathbf{S}^2, S_z] = [S_z, \mathbf{S}^2] = 0$, they are simultaneously measurable.

b) The probability for $\mathbf{S}^2$ being $\frac{3}{4}\hbar^2$ is one. The probability for S_z being $+\hbar/2$ is $(\frac{2}{\sqrt{5}})^2 = \frac{4}{5}$. The probability for S_z being $-\hbar/2$ is $|\frac{i}{\sqrt{5}}|^2 = \frac{1}{5}$.

4. Both x and x^3 contain an *odd number* of raising and lowering operators $A^\dagger$ and A. This, coupled to the fact that states with different n are orthonormal, gives

$$\langle x\rangle = \langle x^3\rangle = 0.$$

As x^2 contains an even number of raising and lowering operators, its expectation value will not be zero

$$\langle x^2\rangle = \langle u_n|x^2|u_n\rangle.$$

From *COLLECTION OF FORMULAE* we get

$$x^2|u_n\rangle = \frac{b^2}{2}(\sqrt{(n+1)(n+2)}|u_{n+2}\rangle + (2n+1)|u_n\rangle + \sqrt{n(n-1)}|u_{n-2}\rangle).$$

The only term that "survives" when we take the scalar product with $\langle u_n|$ is $(2n+1)|u_n\rangle$, so

$$\langle x^2\rangle = \langle u_n|x^2|u_n\rangle = \frac{b^2}{2}(2n+1) = \frac{\hbar}{2m\omega}(2n+1) = \frac{\hbar}{m\omega}(n+\frac{1}{2}).$$

5. Rewrite $L_x^2 + L_y^2 = \mathbf{L}^2 - L_z^2$, which gives

$$H = \frac{\mathbf{L}^2 - L_z^2}{2\hbar^2} + \frac{L_z^2}{3\hbar^2},$$

so

$$HY_{l,m} = (\frac{l(l+1)\hbar^2 - m^2\hbar^2}{2\hbar^2} + \frac{m^2\hbar^2}{3\hbar^2})Y_{l,m},$$

giving

$$E_{l,m} = \frac{l(l+1)}{2} - \frac{m^2}{6}.$$

The lowest (ground state) energy is $E_{0,0} = 0$ (no rotation)!

$l = 1 \Rightarrow m = 0, \pm 1$, gives

$$E_{1,0} = 1\,eV,\ E_{1,\pm 1} = \frac{5}{6}\,eV$$

$l = 2 \Rightarrow m = 0, \pm 1, \pm 2$

$$E_{2,0} = 3\,eV,\ E_{2,\pm 1} = \frac{17}{6}\,eV,\ E_{2,\pm 2} = \frac{7}{3}\,eV$$

and so on.