LULEÅ UNIVERSITY OF TECHNOLOGY
Division of Physics

Course code	MTF107
Examination date	2003-12-19
Time	09.00 - 14.00

Solutions to Quantum Physics

1. i) $Prob = |\psi_1|^2 + |\psi_2|^2 = Prob_1 + Prob_2$

   ii) $Prob = |\psi_1 + \psi_2|^2 = |\psi_1|^2 + |\psi_2|^2 + (\psi_1 + \psi_2)^*(\psi_1 + \psi_2) = Prob_1 + Prob_2 + 2Re(\psi_1 + \psi_2^*)$

   The last term being due to quantum interference, giving rise to both constructive ($2Re(\psi_1 + \psi_2^*) > 0$) and destructive ($2Re(\psi_1 + \psi_2^*) < 0$) interference.

   In i) the neutron acts as a normal, but tiny, "bullet". In ii) the neutron acts as a wave, going through both slits at once!

2. Solve

$$(S_x \cos\theta + S_y \sin\theta)\begin{pmatrix} u \\ v \end{pmatrix} = \frac{\hbar}{2}\lambda\begin{pmatrix} u \\ v \end{pmatrix}.$$

   Using the 2×2 spin matrices S_x, S_y, this gives

$$\begin{pmatrix} 0 & \cos\theta - i\sin\theta \\ \cos\theta + i\sin\theta & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda\begin{pmatrix} u \\ v \end{pmatrix},$$

   giving

$$ve^{-i\theta} = \lambda u$$
$$ue^{i\theta} = \lambda v.$$

   Multiplication of left- and right-hand sides gives

$$uv(\lambda^2 - 1) = 0,$$

   so the eigenvalues are

$$\lambda = \pm 1,$$

   corresponding to spin-up (+1) and spin-down (-1) along the direction given by the operator.

   The eigenstate to $\lambda = +1$ satisfies

$$v = e^{i\theta}u.$$

   Choosing $u = e^{-i\theta/2}$, gives

$$\chi_+ = N\begin{pmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{pmatrix},$$

   where N is the normalization constant, calculated as

$$\chi_+^*\chi_+ = |N|^2(e^{i\theta/2}, e^{-i\theta/2})\begin{pmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{pmatrix} = 2|N|^2,$$

   so

$$\chi_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\theta/2} \\ e^{i\theta/2} \end{pmatrix}.$$

χ_- must be orthogonal to χ_+, giving

$$\chi_- = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\theta/2} \\ -e^{i\theta/2} \end{pmatrix}.$$

[Note: χ_+ and χ_- are defined only up to a constant phase, so

$$\chi_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\theta} \end{pmatrix}$$

and

$$\chi_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -e^{i\theta} \end{pmatrix},$$

are also allowed, etc.]

3. The one-dimensional Schrödinger equation is

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi.$$

Start from

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial t}(\psi^* \psi) = \frac{\partial \psi^*}{\partial t}\psi + \frac{\partial \psi}{\partial t}\psi^*,$$

and use the Schrödinger equation and its complex conjugate to substitute for the time derivatives in $\frac{\partial P}{\partial t}$, giving

$$\frac{\hbar}{2im} \frac{\partial}{\partial x} \left(\frac{\partial \psi^*}{\partial x}\psi - \psi^* \frac{\partial \psi}{\partial x} \right) = -\frac{\partial}{\partial x}\left[\frac{\hbar}{2im}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right)\right],$$

but the quantity in the square bracket is exactly the probability current, j, so

$$\frac{\partial P}{\partial t} = -\frac{\partial j}{\partial x},$$

or

$$\frac{\partial P}{\partial t} + \frac{\partial j}{\partial x} = 0,$$

as was to be shown.

4. Generally, the time dependence of the expectation value of an operator, A, is given by

$$\frac{d}{dt}\langle A \rangle = \langle \frac{\partial A}{\partial t} \rangle + \frac{i}{\hbar}\langle [H, A] \rangle,$$

where A is an arbitrary operator, and H is the Hamiltonian operator.

For our case, this gives

$$\frac{d\langle x \rangle}{dt} = 0 + \frac{i}{\hbar}\langle [H, x] \rangle = \frac{i}{\hbar}\langle [\frac{p^2}{2m}, x] \rangle = \frac{\langle p \rangle}{m}.$$

$$\frac{d\langle p \rangle}{dt} = 0 + \frac{i}{\hbar}\langle [H, p] \rangle = \frac{i}{\hbar}\langle -eE_0 \cos(\omega t)[x, p] \rangle = eE_0 \cos(\omega t).$$

$$\frac{d\langle H \rangle}{dt} = \langle \frac{\partial H}{\partial t} \rangle + 0 = eE_0 \omega \sin(\omega t)\langle x \rangle.$$

5. a) $P = |\psi(x)|^2 = 4\alpha^3 x^2 e^{-2\alpha x}$ peaks when

$$0 = \frac{dP}{dt} = 4\alpha^3 2x(1 - \alpha x)e^{-2\alpha x},$$

that is, when

$$x = \frac{1}{\alpha}.$$

b)

$$\langle x \rangle = \int_0^\infty x(4\alpha^3 x^2 e^{-2\alpha x})dx = \frac{3}{2\alpha}.$$

$$\langle x^2 \rangle = \int_0^\infty x^2(4\alpha^3 x^2 e^{-2\alpha x})dx = \frac{3}{\alpha^2}.$$