LULEÅ UNIVERSITY OF TECHNOLOGY
Division of Physics

Solution to written exam in Quantum Physics F0047T

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1. First choose a coordinate system. If we put the direction of the incoming photon λ along the x-axis positive direction and let the outgoing photon λ' go out along the y-axis in positive direction.

   We can start with the obseravation that as all momentum before the incident is in the positive x-direction this has to be true also after the collision. So as momentum is conserved and the photon λ' leaves in the positive y direction, we make the following conclusions about the electron. The electron must have the same y-momentum in oposite direction to keep the total y momentum zero and the momentum the electron obtaines in the x-driection must be the same as that of the incident photon as the outgoing photon does not carry away any momentum in the x-direction.

   (a) for Compton scattering we have the following relation $\lambda' - \lambda = \frac{h}{m_e c}(1 - cos\theta)$.
   $\lambda = \frac{hc}{E_{photon}} = \frac{6.626\cdot 10^{-34} 2.998\cdot 10^{8}}{100\cdot 10^{3} 1.602\cdot 10^{-19}} = 1.240 \cdot 10^{-11}\text{m} = 0.1240$ Å. The wave lengt of the 'new' photon will be
   $\lambda' = \lambda + \frac{h}{m_e c} = \lambda + \frac{6.626\cdot 10^{-34}}{9.109\cdot 10^{-31} 2.998\cdot 10^{8}} = 1.240 \cdot 10^{-11} + 2.426 \cdot 10^{-12} = 1.4826 \cdot 10^{-11}\text{m}$. The energy is $E = \frac{hc}{\lambda'} = \frac{6.626\cdot 10^{-34} 2.998\cdot 10^{8}}{1.4826\cdot 10^{-11}} = 1.3399 \cdot 10^{-14}\text{m} = 83.637\text{keV} = 83.6$ keV.

   (b) The energy of the electron will be: 100 - 83.6 = 16.4 keV.

   (c) Use conservation of momentum. To calculate the recoil of the electron we have to calculate the momentum of the photon h/λ.

   $$p_x^0 = p_x^1 + p_x^{electron}$$

   $$p_y^0 = p_y^1 + p_y^{electron}$$

   Before the incident $p_x^0 = \frac{6.626\cdot 10^{-34}}{1.240\cdot 10^{-11}} = 5.3435 \cdot 10^{-23}$ kg m/s and $p_y^0 = 0$.
   After the incident the photon has: $p_y^1 = \frac{6.626\cdot 10^{-34}}{1.4826\cdot 10^{-11}} = 4.4692 \cdot 10^{-23}$ kg m/s and $p_x^1 = 0$.
   This yields for the electron $p_x^{electron} = p_x^0 = 5.3435 \cdot 10^{-23}$ kg m/s and
   $p_y^{electron} = -p_y^1 = -4.4692 \cdot 10^{-23}$ kg m/s. The angle of the recoil α is given by
   $\tan\alpha = \frac{p_y^{electron}}{p_x^{electron}} = \frac{-4.4692}{5.3435} = 0.8364$ which gives $\alpha = -39.9^o$ (note sign).
   The length of the electrons momentum vector is
   $p^{electron} = \sqrt{5.3435^2 + 4.4692^2} \cdot 10^{-23} = 6.9661 \cdot 10^{-23}$ kg m/s. The kinetic energy of the electron can also be calculated from
   $E_{kin} = p^2/2m = 6.9661 \cdot 10^{-23}/29.109 \cdot 10^{-31} = 2.6636 \cdot 10^{-15} = 16.6\text{keV}$, the same result as in b.

2. Energinivå för rotationstillstånd J ges av: $E = \frac{\hbar^2}{2I}J(J+1)$, energiskillnad
   $\Delta E = E_{J+1} - E_J = \frac{\hbar^2}{I}(J+1)$ där $I = \mu d^2 = \frac{1}{2}md^2 = \frac{1}{2}14ud^2$. Strålningens våglängd fås ur
   $\Delta E = \frac{hc}{\lambda}$, lös ut J: $J + 1 = \frac{2\pi\frac{1}{2}md^2c}{\hbar\lambda} = \frac{14\ \pi\ 1.66\cdot 10^{-27}(1.094\cdot 10^{-10})^2\ 2.997\cdot 10^{8}}{1.055\cdot 10^{-34}\ 1250\cdot 10^{-6}} \approx 1.98$ dvs J=1.
   Impulsmomentet ges av $\sqrt{J(J+1)}\hbar$ dvs $\hbar\sqrt{6}$ resp $\hbar\sqrt{2}$. Skillnaden blir
   $\hbar(\sqrt{6} - \sqrt{2}) = 1.04\hbar = 1.097\ 10^{-34}$ Nm.

3.

$$< S_x >= \frac{1}{9}(2+i,2)\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2-i \\ 2 \end{pmatrix} = \frac{4}{9}\hbar$$

$$< S_y >= \frac{1}{9}(2+i,2)\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 2-i \\ 2 \end{pmatrix} = \frac{2}{9}\hbar$$

$$< S_z >= \frac{1}{9}(2+i,2)\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2-i \\ 2 \end{pmatrix} = \frac{1}{18}\hbar$$

För spinnmatriserna gäller att σ_i^2 är lika med enhetsmatrisen för i lika med x, y eller z. Detta ger:

$$< S_x^2 >=< S_y^2 >=< S_z^2 >= \hbar^2\frac{1}{36}(2+i,2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2-i \\ 2 \end{pmatrix} = \frac{1}{4}\hbar^2$$

4. Rewrite $L_x^2 + L_y^2 = L^2 - L_z^2$, which gives the Hamiltonian

$$H = \frac{L^2 - L_z^2}{3\hbar^2} + \frac{L_z^2}{4\hbar^2}.$$

The eigenfunctions are $Y_{l,m}$

$$HY_{l,m} = \left(\frac{L^2 - L_z^2}{3\hbar^2} + \frac{L_z^2}{4\hbar^2}\right)Y_{l,m} = \left(\frac{l(l+1)\hbar^2 - m^2\hbar^2}{3\hbar^2} + \frac{m^2\hbar^2}{4\hbar^2}\right)Y_{l,m}.$$

Hence the energies are:

$$E_{l,m} = \left(\frac{l(l+1)}{3} - \frac{m^2}{12}\right).$$

The lowest (ground state) energy is $E_{0,0} = 0$ ($l = 0$ no rotation).

$l = 1 \rightarrow m = 0, \pm 1$, gives $E_{1,0} = \frac{2}{3}\text{eV}$ $E_{1,\pm 1} = \frac{7}{12}\text{eV}$

$l = 2 \rightarrow m = 0, \pm 1, \pm 2$, gives $E_{2,0} = 2\text{eV}$ $E_{2,\pm 1} = \frac{23}{12}\text{eV}$ $E_{2,\pm 2} = \frac{5}{3}\text{eV}$

and so on.

5. The eigenfunctions of the infinite square well in one dimension are (Here a solution of the S.E. in one dimesion is adequate). The width of the well is a.

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a} \quad \text{and the eigenenergys are } E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \quad \text{where} \quad n = 1, 2, 3, ...$$

In three dimensions the eigenfunctions and eigenenergys are (Here an argument about separation of variables is needed to justify the structure of the solution)

$\Psi_{n,m,l}(x,y) = \psi_n(x) \cdot \psi_m(y) \cdot \psi_l(z)$ and eigenenergys $E_{n,m} = E_n + E_m + E_l$ where the indecies are $n = 1, 2, 3, ..$, $m = 1, 2, 3, ..$ and $l = 1, 2, 3, ..$

a) The eigenfunctions inside the box are (note the sidelength is $a/2$ for one of the sides)

$$\Psi_{n,m,l}(x,y,z) = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a} \cdot \sqrt{\frac{2}{a}}\sin\frac{m\pi y}{a} \cdot \sqrt{\frac{4}{a}}\sin\frac{l\pi 2z}{a} \quad \text{where} \quad n = 1, 2, 3, .. , m = 1, 2, 3, .. \text{ and } \quad l = 1, 2, 3,$$

The eigenfunctions outside the box are $\Psi_{n,m,l}(x,y,z) = 0$

b) The seven lowest eigenenergys are (note the 4 associated to the quantum number l this is due to that the length of the box along the z direction is only half of the other two that are of equal length):

$$E_{n,m,l} = \frac{\pi^2\hbar^2}{2ma^2}(n^2 + m^2 + 4l^2), \quad \text{where the 7 lowest are} \quad (n^2 + m^2 + 4l^2) = 6, 9, 12, 14, 18, \text{ and } 21.$$

c) The seven lowest eigenenergys have degeneracys (different ways to choose n, m, l to form the same energy) (either one, two or four) as follows:

$$E_{1,1,1} = \text{ one state } \ (n^2 + m^2 + 4l^2 = 6)$$

$$E_{1,2,1} = E_{2,1,1} = \text{ two states } \ (n^2 + m^2 + 4l^2 = 9)$$
$$E_{2,2,1} = \text{ one state } \ (n^2 + m^2 + 4l^2 = 12)$$
$$E_{1,3,1} = E_{3,1,1} = \text{ two states } \ (n^2 + m^2 + 4l^2 = 14)$$
$$E_{2,3,1} = E_{3,2,1} = \text{ two states } \ (n^2 + m^2 + 4l^2 = 17)$$
$$E_{1,1,2} = \text{ one state } \ (n^2 + m^2 + 4l^2 = 18)$$

Energy number 7 is special as the degeneracy is 4 but all four are not connected through a symmetry operation, ie some of these states are accidentally degenerated. These four can be grouped in the following way.

$$E_{1,2,2} = E_{2,1,2} = \text{ two states } \ (n^2 + m^2 + 4l^2 = 21)$$

$$E_{1,4,1} = E_{4,1,1} = \text{ two states } \ (n^2 + m^2 + 4l^2 = 21)$$