LULEÅ UNIVERSITY OF TECHNOLOGY
Division of Physics

Solution to written exam in QUANTUM PHYSICS F0047T

Examination date: 2013-01-15

1. (a) $i\hbar\frac{\partial^2}{\partial t^2}\cos\omega t = -i\hbar\omega\frac{\partial}{\partial t}\sin\omega t = -i\hbar\omega^2\cos\omega t$ **YES**
   (b) $\frac{\partial}{\partial x}e^{ikx} = ike^{ikx}$ **YES**
   (c) $\frac{\partial}{\partial x}e^{-ax^2} = -2axe^{-ax^2}$ **NO**
   (d) $\frac{\partial}{\partial x}\cos kx = -k\sin kx$ **NO**
   (e) $\frac{\partial}{\partial x}kx = k$ **NO**
   (f) $\hat{P}\sin(kx) = \sin(-kx) = -\sin(kx)$ **YES**

2. Rewrite $L_x^2 + L_y^2 = L^2 - L_z^2$, which gives the Hamiltonian

$$H = \frac{L^2 - L_z^2}{3\hbar^2} + \frac{L_z^2}{4\hbar^2}.$$

The eigenfunctions are $Y_{l,m}$

$$HY_{l,m} = \left(\frac{L^2 - L_z^2}{3\hbar^2} + \frac{L_z^2}{4\hbar^2}\right)Y_{l,m} = \left(\frac{l(l+1)\hbar^2 - m^2\hbar^2}{3\hbar^2} + \frac{m^2\hbar^2}{4\hbar^2}\right)Y_{l,m}.$$

Hence the energies are:

$$E_{l,m} = \left(\frac{l(l+1)}{3} - \frac{m^2}{12}\right).$$

An important issue is the relation between l and m_l, ie $l = 0, 1, 2, 3, ...$ and $m_l = -l, -l+1, ..., 0, l-1, l$. Or it may also be expressed through where it from the treatment is clear how l and m_l are related. The lowest (ground state) energy is $E_{0,0} = 0$ ($l = 0$ no rotation).

$l = 1 \rightarrow m = 0, \pm 1$, gives $E_{1,0} = \frac{2}{3}$eV $E_{1,\pm 1} = \frac{7}{12}$eV

$l = 2 \rightarrow m = 0, \pm 1, \pm 2$, gives $E_{2,0} = 2$eV $E_{2,\pm 1} = \frac{23}{12}$eV $E_{2,\pm 2} = \frac{5}{3}$eV

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3. (a) There are several ways to determine A. One is to integrate and use the normalization condition to solve for A. A different path (done here) is to write the given wave function in terms of eigenfunctions. The eigenfunctions are (PH) $\psi(x) = \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a})$. We can directly conclude that the given wave function consists of eigenfunctions with $n = 1$ and $n = 5$, we can write:

$$\psi(x,0) = \frac{A\sqrt{2}}{\sqrt{2a}}\sin\left(\frac{\pi x}{a}\right) + \frac{\sqrt{2}}{\sqrt{2\cdot 5a}}\sin\left(\frac{5\pi x}{a}\right) = \frac{A}{\sqrt{2}}\psi_1(x,0) + \frac{1}{\sqrt{10}}\psi_5(x,0)$$

As both eigenfunctions are orthonormal the normalisation integral reduces to $\frac{A^2}{2} + \frac{1}{10} = 1$ and hence $A = \sqrt{\frac{18}{10}} = \sqrt{\frac{9}{5}} = \frac{3}{\sqrt{5}}$

(b) The wave function contains only $n = 1$ and $n = 5$ eigenfunctions and therefore the only possible outcomes of an energy meassurement are $E_1 = \frac{\hbar^2\pi^2}{2ma^2}$ with probability $\frac{A^2}{2} = 0.9$ and $E_5 = \frac{\hbar^2\pi^2}{2ma^2}25$ with probability 1 - 0.9 = 0.1 . The average energy is given by $< E >= 0.9E_1 + 0.1E_5 = \frac{\hbar^2\pi^2}{2ma^2}(0.9 + 0.1 \cdot 25) = 3.4 \cdot \frac{\hbar^2\pi^2}{2ma^2} = 1.7 \cdot \frac{\hbar^2\pi^2}{ma^2}$

(c) The time dependent solution is given by $\Psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-iE_nt/\hbar}$ and hence

$$\Psi(x,t) = \sqrt{\frac{9}{10}}\psi_1(x,0)e^{-i\frac{\hbar\pi^2 t}{2ma^2}} + \frac{1}{\sqrt{10}}\psi_5(x,0)e^{-i\frac{25\hbar\pi^2 t}{2ma^2}}$$

4. The harmonic oscillator eigenfunction of the ground state is

$$\psi_0(x) = \sqrt{\frac{\alpha}{\sqrt{\pi}}}\ e^{-\frac{1}{2}\alpha^2x^2} \text{ where } \alpha = \sqrt{\frac{m\omega}{\hbar}}.$$

The four expectation values we are asked to calculate are $\langle x\rangle$, $\langle p\rangle$, $\langle x^2\rangle$, $\langle p^2\rangle$ by explicit integration. By arguments of symmetry we find that $\langle x\rangle = 0$ and the same is for $\langle p\rangle = 0$, as both will be integrals of an odd function that approaches zero exponentially as the arguments go to $\pm\infty$.

The first integral to calculate (use integration by parts) will be for $\langle x^2\rangle$

$$\langle 0 \mid x^2 \mid 0\rangle = \int \psi_0^*(x)x^2\psi_0(x)dx = \int \frac{\alpha}{\sqrt{\pi}}x^2\ e^{-\alpha^2x^2}dx = [\alpha x = y] = \frac{1}{\alpha^2\sqrt{\pi}}\int y^2\ e^{-y^2}dy$$

where the integral taken separatelly will be

$$\int_{-\infty}^{\infty} y^2\ e^{-y^2}dy = [-\frac{y^1}{2}e^{-y^2}]_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\frac{1}{2}e^{-y^2} = 0 + \frac{1}{2}\sqrt{\pi} = \frac{\sqrt{\pi}}{2}$$

and we arrive at:

$$\langle 0 \mid x^2 \mid 0\rangle = \frac{1}{\alpha^2\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = \frac{1}{2\alpha^2}$$

Note on dimensions. As an argument of an exponential function has to be dimensionless this requires the product αx to be dimensionless. As x has dimension 'length' the dimension of α has to be '1/length'. So the expression for $\langle x^2\rangle$ has to contain a one over α squared in order to have the correct dimension.

For the second integral $\langle p^2\rangle$ we have ($p = -i\hbar\frac{\partial}{\partial x}$)

$$\langle 0 \mid p^2 \mid 0\rangle = \int \psi_0^*(x)p^2\psi_0(x)dx = \int \frac{\alpha}{\sqrt{\pi}}e^{-\frac{1}{2}\alpha^2x^2}(-i\hbar\frac{\partial}{\partial x})^2\ e^{-\frac{1}{2}\alpha^2x^2}dx =$$

$$-\hbar^2\int\frac{\alpha}{\sqrt{\pi}}e^{-\frac{1}{2}\alpha^2x^2}\alpha^2(\alpha^2x^2-1)e^{-\frac{1}{2}\alpha^2x^2}dx = -\hbar^2\langle 0 \mid (\alpha^2(\alpha^2x^2-1) \mid 0\rangle =$$

$$-\hbar^2\left(\alpha^2\langle 0 \mid \alpha^2x^2 \mid 0\rangle - \langle 0 \mid \alpha^2 \mid 0\rangle\right) = -\hbar^2\left(\alpha^4\frac{1}{2\alpha^2} - \alpha^2\right) = \frac{1}{2}\hbar^2\alpha^2$$

Uncertainty is defined by: $\langle\Delta p\rangle = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$ and as both $\langle x\rangle$ and $\langle p\rangle$ are zero we arrive at:

$$\langle\Delta p\rangle\langle\Delta x\rangle = \sqrt{\frac{1}{2}\hbar^2\alpha^2\cdot\frac{1}{2\alpha^2}} = \hbar\frac{1}{2} = \frac{\hbar}{2}$$

which is larger or equal to $\frac{\hbar}{2}$ as it should be according to the uncertainty principal.

5. The system is initially in its ground state. The initial state when the particle is under the influence of a potential characterized by the frequency ω_1 is the ground state $\psi_0^{(\omega_1)}$. Immediately after the change to ω_2 we need to analyse the 'new' system with the new eigenfunctions $\psi_j^{(\omega_2)}$. The relation between the 'old' and the 'new' system is given by the completeness relation

$$\psi_0^{(\omega_1)} = \sum_{j=0}^{\infty} c_j \psi_j^{(\omega_2)} \tag{1}$$

where the coefficients c_j describe the spectral distribution for the new eigenstates in relation to the initial. The probability to find the system in the state j is given by $\mid c_j \mid^2$. Here we will use the ground state prior to the sudden change $\psi_0^{(\omega_1)} = \sqrt[4]{\frac{m\omega_1}{\hbar\pi}} e^{-\frac{m\omega_1 x^2}{\hbar 2}}$ and also the ground and first excited state after the change.

In **a** we have to calculate c_0, which is given by the integral:

$$c_0 = \int \left(\psi_0^{(\omega_2)}\right)^* \psi_0^{(\omega_1)} dx \tag{2}$$

The ground state wave function (after) is $\psi_0^{(\omega_2)} = \sqrt[4]{\frac{m\omega_2}{\hbar\pi}} e^{-\frac{m\omega_2 x^2}{\hbar 2}}$. Now calculate c_0 according to

$$c_0 = \int \sqrt[4]{\frac{m\omega_2}{\hbar\pi}} e^{-\frac{m\omega_2 x^2}{\hbar 2}} \sqrt[4]{\frac{m\omega_1}{\hbar\pi}} e^{-\frac{m\omega_1 x^2}{\hbar 2}} dx = \int \sqrt{\frac{m}{\hbar\pi}} \sqrt[4]{\omega_1\omega_2} e^{-\frac{m(\omega_1+\omega_2)x^2}{\hbar 2}} dx \tag{3}$$

Make a change of variables $\sqrt{\frac{m(\omega_1+\omega_2)}{2\hbar}} x = y$ and $dx = \sqrt{\frac{2\hbar}{m(\omega_1+\omega_2)}} dy$.

$$c_0 = \int \sqrt{\frac{m}{\hbar\pi}} \sqrt{\frac{2\hbar}{m(\omega_1+\omega_2)}} \sqrt[4]{\omega_1\omega_2} e^{-y^2} dy = \sqrt[4]{\frac{4\omega_1\omega_2}{(\omega_1+\omega_2)^2}} \tag{4}$$

The probability for the system to be in the new ground state is $\mid c_0 \mid^2 = \sqrt{\frac{4\omega_1\omega_2}{(\omega_1+\omega_2)^2}}. = \frac{2\sqrt{\omega_1\omega_2}}{(\omega_1+\omega_2)}$.

In **b)** we have to make a similar calculation as in **a)**. We can however note that the wave function for the first excited state is $\psi_1^{(\omega_2)} = \sqrt[4]{\frac{m\omega_2}{\hbar\pi}} \sqrt{2} \frac{m\omega_2}{\hbar\pi} x e^{-\frac{m\omega_2 x^2}{\hbar 2}}$. This is however an **odd** function and hence the integrand for c_1 is odd and we arrive at $c_1 = 0.00$