LULEÅ UNIVERSITY OF TECHNOLOGY
Division of Physics

Solution to written exam in QUANTUM PHYSICS F0047T

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1. First choose a coordinate system. Let the direction of the incoming photon λ be along the x-axis's positive direction and let the outgoing photon λ' nearly go out along the y-axis (15 degrees of) in positive direction.

   We can start with the observation that as all momentum before the incident is in the positive x-direction this has to be true also after the collision. So as momentum is conserved and the outgoing photon λ' leaves in the positive y direction, we make the following conclusions about the electron. The electron must have the same y-momentum in opposite direction to keep the total y momentum zero. The momentum the electron obtains in the x-direction has to be the difference between the incident photon and the outgoing photon's momentum in the x-direction.

   (a) for Compton scattering we have the following relation $\lambda' - \lambda = \frac{h}{m_e c}(1-\cos\theta)$.
   $\lambda = \frac{hc}{E_{photon}} = \frac{6.626\cdot 10^{-34} 2.998\cdot 10^{8}}{100\cdot 10^{3} 1.602\cdot 10^{-19}} = 1.240\cdot 10^{-11}\text{m} = 0.1240$ Å. The wave length of the outgoing photon will be
   $\lambda' = \lambda + \frac{h(1-\cos 75)}{m_e c} = \lambda + \frac{6.626\cdot 10^{-34}(1-\cos 75)}{9.109\cdot 10^{-31} 2.998\cdot 10^{8}} = 1.240\cdot 10^{-11} + 1.798\cdot 10^{-12} = 1.4198\cdot 10^{-11}\text{m}$ $= 0.14198$Å. The energy is $E' = \frac{hc}{\lambda'} = \frac{6.626\cdot 10^{-34} 2.998\cdot 10^{8}}{1.4198\cdot 10^{-11}} = 1.3991\cdot 10^{-14}\text{J} = 87.336\text{keV} = 87.3$ keV.
   Another route to the energy may be: $E' = h\nu' = \frac{E}{1+\alpha(1-\cos\theta)}$ where $\alpha = \frac{E}{m_0 c_0^2}$. The dimensionless $\alpha = \frac{100\cdot 10^{3}\cdot 1.602\cdot 10^{-19}}{9.109\cdot 10^{-31}\cdot(2.998\cdot 10^{8})^2} = 0.19567$ and $E' = \frac{100\cdot 10^{3}}{1+0.19567(1-\cos 75)} = 87.3$ keV.

   (b) The energy of the electron will be: 100 - 87.3 = 12.7 keV.

   (c) Use conservation of momentum. To calculate the recoil of the electron we have to calculate the momentum of the photon h/λ.

$$p_x^0 = p_x^1 + p_x^{electron}$$

$$p_y^0 = p_y^1 + p_y^{electron}$$

   Before the incident $p_x^0 = \frac{6.626\cdot 10^{-34}}{1.240\cdot 10^{-11}} = 5.3435\cdot 10^{-23}$ kg m/s and $p_y^0 = 0$.
   After the event the outgoing photon has: $p_y^1 = \frac{6.626\cdot 10^{-34}}{1.4198\cdot 10^{-11}}\sin(75) = 4.5078\cdot 10^{-23}$ kg m/s and $p_x^1 = \frac{6.626\cdot 10^{-34}}{1.4198\cdot 10^{-11}}\cos(75) = 1.2079\cdot 10^{-23}$ kg m/s.
   This yields for the electron $p_x^{electron} = p_x^0 - p_x^1 = (5.3435 - 1.2079)\cdot 10^{-23} = 4.1356\cdot 10^{-23}$ kg m/s and $p_y^{electron} = -p_y^1 = -4.5078\cdot 10^{-23}$ kg m/s. The angle of the recoil α is given by $\tan\alpha = \frac{p_y^{electron}}{p_x^{electron}} = \frac{-4.5078}{4.1356} = -1.0900$ which gives $\alpha = -47.5^o$ (note sign).

   Another way to calculate the angle ϕ of the recoiling electron is: Start with $\cos\theta = \frac{2}{(1+\alpha)^2\tan^2\phi + 1}$ solving for ϕ yields $\tan\phi = \sqrt{\frac{1}{(1+\alpha)^2}\cdot\frac{1+\cos\theta}{1-\cos\theta}}$ and with $\theta = 75$ we arrive at $\tan\phi = 1.089954$ and hence $\phi = 47.46$.

   We can corroborate the result in b) in the following way: The length of the electrons momentum vector is $p^{electron} = \sqrt{4.1356^2 + 4.5078^2}\cdot 10^{-23} = 6.1174\cdot 10^{-23}$ kg m/s. The kinetic energy of the electron can also be calculated from
   $E_{kin} = p^2/2m = (6.1174\cdot 10^{-23})^2/(2\cdot 9.109\cdot 10^{-31}) = 2.0542\cdot 10^{-15}\text{J} = 12.8\text{keV}$, the same result as in b) (well nearly).

2. **a**

The spinor is not normalised and we need to do this first:

$$1 = \chi^* \chi =\mid A \mid^2 (2-5i, 3+i) \begin{pmatrix} 2+5i \\ 3-i \end{pmatrix} =\mid A \mid^2 \mid 2+5i \mid^2 \mid 3-i \mid^2 \rightarrow \ A = \frac{1}{\sqrt{39}}$$

Note an expectation value is always a real number, never a complex one! Even if you had taken A to be a complex number like $A = \frac{i}{\sqrt{39}}$ it would not change the expectation value as the expectation value below only involves $\mid A \mid^2$.

$$< S_x >= \frac{1}{39} (2-5i, 3+i) \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2+5i \\ 3-i \end{pmatrix} = \frac{1}{39}\hbar$$

$$< S_y >= \frac{1}{39} (2-5i, 3+i) \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 2+5i \\ 3-i \end{pmatrix} = -\frac{17}{39}\hbar$$

$$< S_z >= \frac{1}{39} (2-5i, 3+i) \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2+5i \\ 3-i \end{pmatrix} = \frac{19}{78}\hbar$$

b

Measurement along the x direction means: $S = (1,0,0) \cdot (S_x, S_y, S_z) = S_x$. The idea is to expand the initial spinor χ into the eigenspinors of S_x. So we start to calculate the eigenvalues and eigenspinors to S_x. The spin operator S_x is

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we find the eigenvalues from the following equation

$$S_n \chi = \lambda \chi \Leftrightarrow \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix} \tag{1}$$

We find the eigenvalues from the equation

$$\begin{vmatrix} -\lambda & 1\frac{\hbar}{2} \\ 1\frac{\hbar}{2} & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda = \pm\frac{\hbar}{2}$$

The eigenspinors to S_x corresponding to the $+\frac{\hbar}{2}$ we get from

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = +\frac{\hbar}{2} \begin{pmatrix} a \\ b \end{pmatrix}$$

The two equations above are linearly dependent and one of them is

$$a = b \Leftrightarrow \text{ let } \ b = 1 \ \text{ and hence } \ a = 1$$

This gives the unnormalised spinor

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and after normalisation we have} \quad \chi_{x+} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The other eigenspinor χ_{x-} has to be orthogonal to χ_{x+}. An appropriate choice is:

$$\chi_{x-} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Now we can expand the initial spinor χ in these eigenspinors to S_x.

$$\chi = \frac{1}{\sqrt{39}} \begin{pmatrix} 2+5i \\ 3-i \end{pmatrix} = b_+ \chi_{x+} + b_- \chi_{x-}$$

The coefficient b_+ is given by

$$b_+ = \chi_{x+}^* \chi = \frac{1}{\sqrt{78}} (1 \quad 1) * \begin{pmatrix} 2+5i \\ 3-i \end{pmatrix} = \frac{1}{\sqrt{78}} (2+5i+3-i) = \frac{1}{\sqrt{78}} (5+4i)$$

A similar calculation gives b_- :

$$b_- = \chi_{x+}^* \chi = \frac{1}{\sqrt{78}} (1 \quad -1) * \begin{pmatrix} 2+5i \\ 3-i \end{pmatrix} = \frac{1}{\sqrt{78}} (2+5i-3+i) = \frac{1}{\sqrt{78}} (-1+6i)$$

We may now check that $\mid b_+ \mid^2 + \mid b_- \mid^2 = 1$

$$\mid b_+ \mid^2 + \mid b_- \mid^2 = \frac{1}{78} (25+16+1+36) = 1 \quad \text{ok}$$

The probability (to get $+\frac{\hbar}{2}$) is given by $|b_+|^2$.

$$|b_+|^2 = \frac{1}{78} (25+16) = \frac{41}{78} \approx \mathbf{0.526}$$

and (to get $-\frac{\hbar}{2}$) is given by $|b_-|^2$.

$$|b_-|^2 = \frac{1}{78} (1+36) = \frac{37}{78} \approx \mathbf{0.474}$$

You may make the following check for consistency:

$$< S_x > = \left(\frac{41}{78} (\frac{\hbar}{2}) + \frac{37}{78} (-\frac{\hbar}{2}) \right) = \frac{1}{39} \hbar$$

The same result as in part **a**.

3. Rewrite the wave function in terms of spherical harmonics: (polar coordinates: $x = r \sin\theta \sin\phi, z = r\cos\theta$ and hence $zx = r^2 \cos\theta \sin\theta (e^{i\phi} + e^{-i\phi})/2$ using the Euler relations) the appropriate spherical harmonics can now be identified and we arrive at

$$\psi(\boldsymbol{r}) = \psi(x,y,z) = N(zx) e^{-r/3a_0} = N \frac{r^2}{2} \sqrt{\frac{8\pi}{15}} (-Y_{2,1} + Y_{2,-1}) e^{-r/3a_0} \ . \tag{2}$$

As all the involved $Y_{l,m}$ have $l = 2$ the probability to get $\mathbf{L}^2 = 2(2+1)\hbar^2 = 6\hbar^2$ is one. For the operator L_z we note the two spherical harmonics have the same pre factor (one has -1 and the other has +1 but the absolute value square is the same) ie they will have the same probability. The probability to find $m = 2\hbar$ is 0, for $m = 1\hbar$ is $\frac{1}{2}$, for $m = 0\hbar$ is 0 for $m = -1\hbar$ is $\frac{1}{2}$, and for $m = -2\hbar$ is 0. As all the involved $Y_{l,m}$ have $l = 2$ the probability to get $\mathbf{L}^2 = 2(2+1)\hbar^2 = 6\hbar^2$ is one.

b. To calculate the expectation value $< r >$ we need to normalise the given wave function if we wish to do the integral. In order to achieve this in a simple way is to identify the radial wave function. As l is equal to 2 we know that n cannot be equal to 1 or 2 it has to be **larger** or **equal** to 3. By inspection of eq (2) and 2 we find $n = 3$ this function has the correct exponential and the correct power of r (r^2) and hence $R_{3,2}(r) = \frac{2\sqrt{2}}{27\sqrt{5}}\left(\frac{Z}{3a_0}\right)^{3/2}\left(\frac{Zr}{a_0}\right)^2 e^{-Zr/3a_0}$. We also note that $Y_{2,1}$ and $Y_{2,-1}$ are normalised but the sum $(-Y_{2,1} + Y_{2,-1})$ is not normalised. The sum has to be changed to $(-\frac{1}{\sqrt{2}}Y_{2,1} + \frac{1}{\sqrt{2}}Y_{2,-1})$ in order to be normalised. Note that $R_{3,2}(r)$ contains an r^2 term as also a $e^{-r/3a_0}$ term. The wave function can now be completed to the following normalized wave function (note that we do not need to calculate the constant N as all separate parts of $\psi(r)$ are normalised by them selves)

$$\psi(\boldsymbol{r}) = \psi(x, y, z) = N(zx)e^{-r/3a_0} = R_{3,2}(r)(-\frac{1}{\sqrt{2}}Y_{2,1} + \frac{1}{\sqrt{2}}Y_{2,-1})$$

From physics handbook page 292 you find

$$\langle r\rangle = \frac{1}{2}\left[3n^2 - l(l+1)\right]\left(\frac{a_0}{Z}\right) = \frac{1}{2}\left[3\ 3^2 - 2(2+1)\right]\left(\frac{a_0}{1}\right) = \frac{21}{2}a_0 =$$

$$10.5\ \cdot 0.5292\ \text{Å} = 5.56\ \text{Å}.$$

You may also do the integral directly like this:

$$\langle r\rangle = \int_0^\infty\int_0^\pi\int_0^{2\pi} d\phi\ d\theta\ dr\ r^2\sin(\theta)\ r\ \mid R_{3,2}(r)\mid^2 \mid \left(-\frac{1}{\sqrt{2}}Y_{2,1} + \frac{1}{\sqrt{2}}Y_{2,-1}\right)\mid^2 =$$

$$\int_0^\infty dr\ r^3 \mid R_{3,2}(r)\mid^2 = \frac{21}{2}a_0 = 10.5\ \cdot 0.5292\ \text{Å} = 5.56\ \text{Å}.$$

4. This is a 2 dimensional problem with a Schrödinger equation (where $V(x, y) = 0$) like

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi(x,y) - \frac{\hbar^2}{2m}\frac{d^2}{dy^2}\Psi(x,y) = E\Psi(x,y)$$

This equation is separable and the ansatz $\Psi(x, y) = \psi(x) * \psi(y)$ gives the following result

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_x(x) - \frac{\hbar^2}{2m}\frac{d^2}{dy^2}\psi_y(y) = E_x\psi_x(x) + E_y\psi_y(y)$$

ie two independent one dimensional Schrödinger equations one for the variable x and on for y. We therefor solve the one dimensional problem first and after that we construct the two dimensional solution. To find the eigenfunctions we need to solve the Schrödinger equation which is (in the region where $V(x)$ is zero)

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi = E\Psi \ \rightarrow\ \frac{d^2}{dx^2}\Psi + k^2\Psi = 0 \text{ where } k^2 = \frac{2mE}{\hbar^2}$$

Solutions are of the kind:

$$\Psi(x) = A\cos kx + B\sin kx$$

Now we need to take the boundary conditions for the wave function Ψ $\left(\Psi(-\frac{a}{2}) = \Psi(\frac{a}{2}) = 0\right)$ into account.

$$A\cos(-\frac{ka}{2}) + B\sin(-\frac{ka}{2}) = 0 \text{ and } A\cos(\frac{ka}{2}) + B\sin(\frac{ka}{2}) = 0$$

Adding the two conditions gives: $\cos(\frac{ka}{2}) = 0$ and subtracting them gives $\sin(\frac{ka}{2}) = 0$. These two conditions cannot be fulfilled at the same time, so either A or B has to be zero. We start with $A = 0$ and we get the following solution: The normalising constant $B = \sqrt{\frac{2}{a}}$ you get from the condition $\int_{-a/2}^{a/2} \mid \Psi \mid^2 dx = 1$. The condition $\sin(\frac{ka}{2}) = 0$ gives $\frac{ka}{2} = \frac{\pi}{2} * (even - integer)$. The solution is:

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a}) \text{ with eigenenergys } E_n = \frac{n^2\pi^2\hbar^2}{2Ma^2} \quad \text{where} \quad n = 2, 4, 6, ... \tag{3}$$

In a similar way the other function is analysed $(A = 0)$ which gives: The condition $\cos(\frac{ka}{2}) = 0$ gives $\frac{ka}{2} = \frac{\pi}{2} * (odd - integer)$. The solution is:

$$\psi_n(x) = \sqrt{\frac{2}{a}}\cos(\frac{n\pi x}{a}) \text{ with eigenenergys } E_n = \frac{n^2\pi^2\hbar^2}{2Ma^2} \quad \text{where} \quad n = 1, 3, 5, ... \tag{4}$$

The eigenfunctions in the y direction are the same as for the x direction as the potential is similar for this direction. Now we have the eigenfunctions of the one dimensional problem and the solution to the 2 dimensional problem is readily produced. The eigenfunctions are:

$$\Psi_{n,m}(x, y) = \psi_n(x) \cdot \psi_m(y) \text{ eigenenergys } E_{n,m} = E_n + E_m \text{where } n = 1, 2, ,. \text{ and } m = 1, 2, ,. \tag{5}$$

In the area where the potential is infinite the wave function is equal to zero.

An **alternative route** taken by many students has been to present a calculation with the following boundary conditions: Ψ $(\Psi(0) = \Psi(a) = 0)$ into account. In this case the solution is for these boundary conditions:

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a}) \text{ with eigenenergys } E_n = \frac{n^2\pi^2\hbar^2}{2Ma^2} \quad \text{where} \quad n = 1, 2, 3, ... \tag{6}$$

This solution has to be adapted to the boundary conditions related to this exam problem:

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin(\frac{n\pi}{a}(x + \frac{a}{2})) \text{ with eigenenergys } E_n = \frac{n^2\pi^2\hbar^2}{2Ma^2} \quad \text{where} \quad n = 1, 2, 3, ... \tag{7}$$

$\psi_n(x) = \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a} + \frac{n\pi}{2}) = \sqrt{\frac{2}{a}}\left(\sin(\frac{n\pi x}{a}) \cdot \cos(\frac{n\pi}{2}) + \cos(\frac{n\pi x}{a}) \cdot \sin(\frac{n\pi}{2})\right)$. We see that we recover the solution in eq (3), (4) and (5) as we let n run from 1 to ∞.

b) Now we turn to the question of **parity**, ie whether the wave function is *odd* or *even* under a change of coordinates from (x, y) to $(-x, -y)$. The one dimensional eigenfunctions in eq (3) and (4) have a definite parity. The functions in (3) are odd whereas the functions in (4) are even. As the eigenstates for the 2 dimensional system are formed from eq (5) ie products of functions that are even or odd the total function itself will be either even or odd as well.

The four lowest eigenenergies are given by

$$E_{n,m} = \frac{\pi^2\hbar^2}{2Ma^2}(n^2+m^2), \quad \text{where the 4 lowest are} \quad (n^2+m^2) = 2, 5, 8, 10.$$

When we form the eigenstates we need to keep track of the parity of the $\psi_n(x)$ and $\psi_m(y)$. It is therefore necessary to have the functions in the form like in eq (3) and (4) to identify the parity as odd or even. This is difficult if you try with functions like eq (7) even though it is a correct eigenstate it is hard to identify their parity.

$$E_{1,1} = \text{ one state } (n^2+m^2=2) \qquad \textbf{even * even} = \text{even}$$

$$E_{1,2} = E_{2,1} = \text{ two states } (n^2+m^2=5) \qquad \text{odd * even} = \text{odd}$$

$$E_{2,2} = \text{ one state } (n^2+m^2=8) \qquad \textbf{odd * odd} = \text{even}$$

$$E_{1,3} = E_{3,1} = \text{ two states } (n^2+m^2=10) \qquad \textbf{even * even} = \text{even}$$

So of the four energys (states) only one is **odd** and three are **even**.

5. The task is to calculate the change of the difference between two energy levels (ground state E_0 and first excited state E_1) for a harmonic oscillator due to a perturbation H^1 to the potential.

$$E_1^1 - E_0^1 = E_1 + \langle 1 \mid H^1 \mid 1\rangle - \Big(E_0 + \langle 0 \mid H^1 \mid 0\rangle\Big)$$

The two harmonic oscillator eigenfunctions that are of interest are :

$$\psi_0(x) = \sqrt{\frac{\alpha}{\sqrt{\pi}}}\; e^{-\frac{1}{2}\alpha^2x^2} \text{ and } \psi_1(x) = \sqrt{\frac{\alpha}{2\sqrt{\pi}}}2\alpha x\; e^{-\frac{1}{2}\alpha^2x^2} \text{ where } \alpha = \sqrt{\frac{m\omega}{\hbar}}$$

The first integral to calculate (use integration by parts) will be for the change of the ground state energy

$$\langle 0 \mid H^1 \mid 0\rangle = \int \psi_0^*(x)H^1\psi_0(x)dx = \int \frac{\alpha}{\sqrt{\pi}}Ax^4\; e^{-\alpha^2x^2}dx = [\alpha x = y] = \frac{A}{\alpha^4\sqrt{\pi}}\int y^4\; e^{-y^2}dy$$

where the integral taken separately will be

$$\int_{-\infty}^{\infty} y^4\; e^{-y^2}dy = [-\frac{y^3}{2}e^{-y^2}]_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\frac{3y^2}{2}e^{-y^2} = [-\frac{3y^1}{4}e^{-y^2}]_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\frac{3}{4}e^{-y^2} = \frac{3}{4}\sqrt{\pi}$$

Hence the shift of the ground state energy will be

$$\langle 0 \mid H^1 \mid 0\rangle = \frac{A}{\alpha^4\sqrt{\pi}}\frac{3}{4}\sqrt{\pi} = \frac{3A}{4\alpha^4} = \frac{3A}{4}\left(\frac{\hbar}{m\omega}\right)^2$$

The second integral to calculate (use integration by parts) will be for the change of the energy of the lowest excited state.

$$\langle 1 \mid H^1 \mid 1\rangle = \int \psi_1^*(x)H^1\psi_1(x)dx = \int \frac{\alpha}{2\sqrt{\pi}}Ax^4\; 4\alpha^2x^2e^{-\alpha^2x^2}dx = [\alpha x = y] = \frac{4A}{\alpha^4\sqrt{\pi}}\int y^6\; e^{-y^2}dy$$

where the integral taken separately will be

$$\int_{-\infty}^{\infty} y^6 \ e^{-y^2} dy = [-\frac{y^5}{2} e^{-y^2}]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{5y^4}{2} e^{-y^2} = [-\frac{5y^3}{4} e^{-y^2}]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{15y^2}{4} e^{-y^2} =$$

$$= [-\frac{15y^1}{8} e^{-y^2}]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{15}{8} e^{-y^2} = \frac{15}{8}\sqrt{\pi}$$

Hence the shift of the energy of the lowest excited state will be

$$\langle 1 \mid H^1 \mid 1 \rangle = \frac{4A}{\alpha^4 \sqrt{\pi}} \frac{15}{8} \sqrt{\pi} = \frac{15A}{2\alpha^4} = \frac{15A}{2} \left(\frac{\hbar}{m\omega} \right)^2$$

The difference in the perturbed energys will be

$$E_1^1 - E_0^1 = \frac{3}{2}\hbar\omega + \frac{15A}{2\alpha^4} - (\frac{1}{2}\hbar\omega + \frac{3A}{4\alpha^4}) = \hbar\omega + \frac{27A}{4\alpha^4} = \hbar\omega + \frac{27A}{4} \left(\frac{\hbar}{m\omega} \right)^2$$

Note that the constant A has dimension.