LULEÅ UNIVERSITY OF TECHNOLOGY
Division of Physics

Solution to written exam in Quantum Physics F0047T

Examination date: 2015-08-29

The solutions are just suggestions. They may contain several alternative routes.

1. (a) There are several ways to determine A. One is to integrate and use the normalization condition to solve for A. A different path (done here) is to write the given wave function in terms of eigenfunctions (here particle in a box). The eigenfunctions are (PH) $\psi_n(x) = \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a})$. We can directly conclude that the given wave function consists of $n = 1$, $n = 5$ and $n = 7$ functions, we can write:

$$\psi(x,0) = \frac{\sqrt{11}}{\sqrt{8\cdot 2}}\frac{\sqrt{2}}{\sqrt{a}}\sin\left(\frac{\pi x}{a}\right) + \frac{\sqrt{2}}{2\sqrt{2\cdot a}}\sin\left(\frac{4\pi x}{a}\right) + \frac{A\sqrt{2}}{\sqrt{2\cdot a}}\sin\left(\frac{5\pi x}{a}\right) =$$

$$\frac{\sqrt{11}}{\sqrt{16}}\psi_1(x,0) + \frac{1}{\sqrt{8}}\psi_4(x,0) + \frac{A}{\sqrt{2}}\psi_5(x,0)$$

As all three eigenfunctions are orthonormal the normalisation integral reduces to $\frac{11}{16} + \frac{1}{8} + \frac{A^2}{2} = 1$ and hence $A = \sqrt{\frac{3}{8}}$ (≈ 0.612).

(b) The wave function contains only $n = 1$, $n = 4$ and $n = 5$ eigenfunctions and therefore the only possible outcome of an energy meassurement are $E_1 = \frac{\hbar^2\pi^2}{2ma^2}$ with probability $\frac{11}{16}$ and $E_4 = \frac{\hbar^2\pi^2}{2ma^2}16$ with probability $\frac{1}{8}$ and $E_5 = \frac{\hbar^2\pi^2}{2ma^2}25$ with probability $\frac{A^2}{2} = \frac{3}{16}$.
The average energy is given by
$<E>= \frac{11}{16}E_1 + \frac{1}{8}E_4 + \frac{3}{16}E_5 = \frac{\hbar^2\pi^2}{2ma^2}(\frac{11}{16} + \frac{1}{8}\cdot 16 + \frac{3}{16}\cdot 25) = \frac{118}{16}\cdot\frac{\hbar^2\pi^2}{2ma^2} = \frac{59}{8}\cdot\frac{\hbar^2\pi^2}{2ma^2}$

(c) The time dependent solution is given by $\Psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-iE_nt/\hbar}$ and hence

$$\Psi(x,t) = \sqrt{\frac{11}{16}}\psi_1(x,0)e^{-i\frac{\hbar\pi^2 t}{2ma^2}} + \frac{1}{\sqrt{8}}\psi_4(x,0)e^{-i\frac{16\hbar\pi^2 t}{2ma^2}} + \sqrt{\frac{3}{16}}\psi_5(x,0)e^{-i\frac{25\hbar\pi^2 t}{2ma^2}}$$

2. Rewrite $L_x^2 + L_y^2 = L^2 - L_z^2$, which gives the Hamiltonian

$$H = \frac{L^2 - L_z^2}{2\hbar^2} + \frac{L_z^2}{3\hbar^2}.$$

The eigenfunctions are $Y_{l,m}$

$$HY_{l,m} = \left(\frac{L^2 - L_z^2}{2\hbar^2} + \frac{L_z^2}{3\hbar^2}\right)Y_{l,m} = \left(\frac{l(l+1)\hbar^2 - m^2\hbar^2}{2\hbar^2} + \frac{m^2\hbar^2}{3\hbar^2}\right)Y_{l,m}.$$

Hence the energies are:

$$E_{l,m} = \left(\frac{l(l+1)}{2} - \frac{m^2}{6}\right).$$

The lowest (ground state) energy is $E_{0,0} = 0$ ($l = 0$ no rotation).

$l = 1 \to m = 0, \pm 1$, gives $E_{1,0} = 1\text{eV}$ $E_{1,\pm 1} = \frac{5}{6}\text{eV}$

$l = 2 \to m = 0, \pm 1, \pm 2$, gives $E_{2,0} = 3\text{eV}$ $E_{2,\pm 1} = \frac{17}{6}\text{eV}$ $E_{2,\pm 2} = \frac{7}{3}\text{eV}$

and so on.

3. A measurement of the spin in the direction $\hat{n} = \sin(\frac{\pi}{4})\hat{e}_y + \cos(\frac{\pi}{4})\hat{e}_z = \frac{1}{\sqrt{2}}\hat{e}_y + \frac{1}{\sqrt{2}}\hat{e}_z$. The spin operator $S_{\hat{n}}$ is

$$S_{\hat{n}} = \frac{1}{\sqrt{2}}S_y + \frac{1}{\sqrt{2}}S_z = \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}$$

The eigenvalue equation is

$$S_{\hat{n}}\chi = \lambda\chi \Leftrightarrow \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix} \tag{1}$$

We find the eigenvalues from

$$\begin{vmatrix} \frac{\hbar}{2\sqrt{2}} - \lambda & -i\frac{\hbar}{2\sqrt{2}} \\ i\frac{\hbar}{2\sqrt{2}} & -\frac{\hbar}{2\sqrt{2}} - \lambda \end{vmatrix} = 0 \Rightarrow \lambda = \pm\frac{\hbar}{2}$$

The eigenspinors to S_n corresponding to the $+\frac{\hbar}{2}$ we get from

$$\frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = +\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\frac{a}{\sqrt{2}} - \frac{ib}{\sqrt{2}} = a \Leftrightarrow a(\sqrt{2}-1) = -ib \ \text{ let } \ b = 1 \ \text{ and hence } \ a = \frac{-i}{\sqrt{2}-1}$$

This gives the unnormalised spinor

$$\begin{pmatrix} -\frac{i}{\sqrt{2}-1} \\ 1 \end{pmatrix} \quad \text{and after normalisation we have} \quad \chi_{\hat{n}+} = \frac{1}{\sqrt{2(2+\sqrt{2})}}\begin{pmatrix} -\frac{i}{\sqrt{2}-1} \\ 1 \end{pmatrix}$$

Now we can expand the initial eigenspinor χ_+ in these eigenspinors to S_n, the second eigenspinor you can get from orthogonality to the first one.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = A\frac{1}{\sqrt{2(2+\sqrt{2})}}\begin{pmatrix} -\frac{i}{\sqrt{2}-1} \\ 1 \end{pmatrix} + B\frac{1}{\sqrt{2(2+\sqrt{2})}}\begin{pmatrix} 1 \\ \frac{-i}{\sqrt{2}-1} \end{pmatrix}$$

The coefficients are subjected to the normalisation condition $|A|^2 + |B|^2 = 1$. The coefficient A can be obtained by multiplying the previous equation from the left with $\chi^*_{\hat{n}+}$.

$$A = \frac{1}{\sqrt{2(2+\sqrt{2})}}\begin{pmatrix} -\frac{i}{\sqrt{2}-1} & 1 \end{pmatrix} * \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{i}{\sqrt{2}-1}\cdot\frac{1}{\sqrt{2(2+\sqrt{2})}}$$

The probability (to get $+\frac{\hbar}{2}$) is given by $|A|^2$.

$$|A|^2 = \frac{3+2\sqrt{2}}{4+2\sqrt{2}} = 0.8535533906$$

and (to get $-\frac{\hbar}{2}$) for $|B|^2$.

$$|B|^2 = \frac{1}{4+2\sqrt{2}} = 0.1464466094$$

To find the probability for $+\frac{\hbar}{2}$ in the z-direction for the up state of S_n express the state in the eigenspinors to S_z.

$$\chi_{\hat{n}+} = \frac{1}{\sqrt{2(2+\sqrt{2})}}\begin{pmatrix} -\frac{i}{\sqrt{2}-1} \\ 1 \end{pmatrix} = -\frac{i}{\sqrt{2}-1}\cdot\frac{1}{\sqrt{2(2+\sqrt{2})}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2(2+\sqrt{2})}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The probability is given by the square of the coefficient:

$$\left| -\frac{i}{\sqrt{2}-1} \cdot \frac{1}{\sqrt{2(2+\sqrt{2})}} \right|^2 = 0.8535533906$$

4. The eigenfunctions of the infinite square well in one dimension are (Here a solution of the S.E. in one dimesion is adequate). The width of the well is a.

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \quad \text{and the eigenenergys are } E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \quad \text{where} \quad n = 1, 2, 3, ...$$

In three dimensions the eigenfunctions and eigenenergys are (Here an argument about separation of variables is needed to justify the structure of the solution)

$\Psi_{n,m,l}(x, y) = \psi_n(x) \cdot \psi_m(y) \cdot \psi_l(z)$ and eigenenergys $E_{n,m} = E_n + E_m + E_l$ where the indecies are $n = 1, 2, 3, ..$, $m = 1, 2, 3, ..$ and $l = 1, 2, 3, ..$

a) The eigenfunctions inside the box are (note the sidelength is $a/2$ for one of the sides)

$$\Psi_{n,m,l}(x, y, z) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \cdot \sqrt{\frac{2}{a}} \sin \frac{m\pi y}{a} \cdot \sqrt{\frac{4}{a}} \sin \frac{l\pi 2z}{a} \quad \text{where} \quad n = 1, 2, 3, .. , m = 1, 2, 3, .. \text{ and } \quad l = 1, 2, 3,$$

The eigenfunctions outside the box are $\Psi_{n,m,l}(x, y, z) = 0$

b) The seven lowest eigenenergys are (note the 4 associated to the quantum number l this is due to that the length of the box along the z direction is only half of the other two that are of equal length):

$$E_{n,m,l} = \frac{\pi^2\hbar^2}{2ma^2}(n^2 + m^2 + 4l^2), \quad \text{where the 7 lowest are} \quad (n^2 + m^2 + 4l^2) = 6, 9, 12, 14, 18, \text{ and } 21.$$

c) The seven lowest eigenenergys have degeneracys (different ways to choose n, m, l to form the same energy) (either one, two or four) as follows:

$$E_{1,1,1} = \text{ one state } \ (n^2 + m^2 + 4l^2 = 6)$$
$$E_{1,2,1} = E_{2,1,1} = \text{ two states } \ (n^2 + m^2 + 4l^2 = 9)$$
$$E_{2,2,1} = \text{ one state } \ (n^2 + m^2 + 4l^2 = 12)$$
$$E_{1,3,1} = E_{3,1,1} = \text{ two states } \ (n^2 + m^2 + 4l^2 = 14)$$
$$E_{2,3,1} = E_{3,2,1} = \text{ two states } \ (n^2 + m^2 + 4l^2 = 17)$$
$$E_{1,1,2} = \text{ one state } \ (n^2 + m^2 + 4l^2 = 18)$$

Energy number 7 is special as the degeneracy is 4 but all four are not connected through a symmetry operation, ie some of these states are accidentally degenerated. These four can be grouped in the following way.

$$E_{1,2,2} = E_{2,1,2} = \text{ two states } \ (n^2 + m^2 + 4l^2 = 21)$$
$$E_{1,4,1} = E_{4,1,1} = \text{ two states } \ (n^2 + m^2 + 4l^2 = 21)$$

5. The task is to calculate the change of the energy levels (ground state E_0 and first excited state E_1) for a harmonic oscillator due to a perturbation H^1 to the potential.

The two harmonic oscillator eigenfunctions that are of interest are :

$$\psi_0(x) = \sqrt{\frac{\alpha}{\sqrt{\pi}}}\, e^{-\frac{1}{2}\alpha^2 x^2} \quad \text{and} \quad \psi_1(x) = \sqrt{\frac{\alpha}{2\sqrt{\pi}}} 2\alpha x\, e^{-\frac{1}{2}\alpha^2 x^2} \quad \text{where} \quad \alpha = \sqrt{\frac{m\omega}{\hbar}}$$

(a) Here we have a perturbation γx^4 where γ is small in some sence. The first integral to calculate (use integration by parts) will be for the change of the ground state energy

$$\langle 0 \mid \gamma x^4 \mid 0 \rangle = \int \psi_0^*(x) \gamma x^4 \psi_0(x) dx = \int \frac{\alpha}{\sqrt{\pi}} \gamma x^4\, e^{-\alpha^2 x^2} dx = [\alpha x = y] = \frac{\gamma}{\alpha^4 \sqrt{\pi}} \int y^4\, e^{-y^2} dy$$

where the integral taken separately will be

$$\int_{-\infty}^{\infty} y^4\, e^{-y^2} dy = [-\frac{y^3}{2} e^{-y^2}]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{3y^2}{2} e^{-y^2} = [-\frac{3y^1}{4} e^{-y^2}]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{3}{4} e^{-y^2} = \frac{3}{4}\sqrt{\pi}$$

Hence the shift of the ground state energy will be

$$\langle 0 \mid \gamma x^4 \mid 0 \rangle = \frac{\gamma}{\alpha^4 \sqrt{\pi}} \frac{3}{4} \sqrt{\pi} = \frac{3\gamma}{4\alpha^4} = \frac{3\gamma}{4} \left(\frac{\hbar}{m\omega} \right)^2$$

The energy of the unperturbed groundstate is $E_0 = \frac{\hbar\omega}{2}$. Hence the energy of the perturbed groundstate is

$$E_0^{\text{perturbed}} = \frac{\hbar\omega}{2} + \frac{3\gamma}{4} \left(\frac{\hbar}{m\omega} \right)^2$$

(b) Here we have a perturbation ϵx where ϵ is small in some sence. The integrals to be calculated are $\langle 0 \mid \epsilon x \mid 0 \rangle$ and $\langle 1 \mid \epsilon x \mid 1 \rangle$. The squares of both eigenfunctions are even functions and as the perturbation is odd both integrals will be zero.

Hence there is no change in energy to first order.