LULEÅ UNIVERSITY OF TECHNOLOGY
Division of Physics

Solution to written exam in Quantum Physics and Statistical Physics MTF131

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1. The eigenfunctions of the infinite square well in one dimension are (Here a solution of the S.E. in one dimesion is adequate)

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \quad \text{and the eigenenergys are } E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \quad \text{where} \quad n = 1, 2, 3, ...$$

In two dimensions the eigenfunctions and eigenenergys are (Here an argument about separation of variables is needed)

$\Psi_{n,m}(x, y) = \psi_n(x)\cdot\psi_m(y)$ and eigenenergys $E_{n,m} = E_n + E_m$ where $n = 1, 2, 3, ..$ and $m = 1, 2, 3, ..$

a) The eigenfunctions inside the square

$$\Psi_{n,m}(x, y) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \cdot \sqrt{\frac{2}{a}} \sin \frac{m\pi y}{a} \quad \text{where} \quad n = 1, 2, 3, .. \text{ and } m = 1, 2, 3, ..$$

The eigenfunctions outside the square $\Psi_{n,m}(x, y) = 0$
b) The four lowest eigenenergys are

$$E_{n,m} = \frac{\pi^2\hbar^2}{2ma^2}(n^2 + m^2), \quad \text{where the 4 lowest are} \quad (n^2 + m^2) = 2, 5, 8, \text{ and } 10.$$

c) The four lowest eigenenergys have degeneracys (either one or two) as follows:

$$E_{1,1} = \text{ one state } (n^2 + m^2 = 2)$$
$$E_{1,2} = E_{2,1} = \text{ two states } (n^2 + m^2 = 5)$$
$$E_{2,2} = \text{ one state } (n^2 + m^2 = 8)$$
$$E_{1,3} = E_{3,1} = \text{ two states } (n^2 + m^2 = 10)$$

2. The proton has 2 possible spin directions - and + with energys $-Bm_\mu$ and $+Bm_\mu$. The partition function for a proton in the magnetic field B is

$$Z = e^{Bm_\mu/\tau} + e^{-Bm_\mu/\tau}.$$

The probabillity for a proton to be in the - direction (or state) is given by

$$P(-Bm_\mu) = \frac{e^{Bm_\mu/\tau}}{e^{Bm_\mu/\tau} + e^{-Bm_\mu/\tau}}.$$

and similar for the + direction. As there are N protons the number of protons in the - direction will be $N_- = NP(-Bm_\mu)$ and in the + direction $N_+ = NP(+Bm_\mu)$. The absorbed power is proportional to the difference in the number of protons in the + state to the number in the - state, Power $\propto N_+ - N_- = N(P(+Bm_\mu) - P(-Bm_\mu)$.

$$N_- - N_+ = N\frac{e^{Bm_\mu/\tau} - e^{-Bm_\mu/\tau}}{e^{Bm_\mu/\tau} + e^{-Bm_\mu/\tau}} = N \tanh(e^{Bm_\mu/\tau})$$

In the high temperature limit ($Bm_\mu << \tau$) we have

$$N_- - N_+ = N \tanh(e^{Bm_\mu/\tau}) \approx N\frac{Bm_\mu}{\tau}$$

3. Rewrite the wave function in terms of spherical harmonics:

$$\psi(\boldsymbol{r}) = \psi(x,y,z) = N(xy+yz+zx)e^{-\alpha r} =$$

$$= Nr^2e^{-\alpha r}\left(\frac{1}{4i}\sqrt{\frac{32\pi}{15}}(Y_{2,2}-Y_{2,-2}) + \frac{1-i}{2}\sqrt{\frac{8\pi}{15}}(Y_{2,1} + \frac{1+i}{2}\sqrt{\frac{8\pi}{15}}(Y_{2,-1})\right)$$

As all the $Y_{l,m}$ have $l=2$ the probability to get $\mathbf{L}^2 = 2(2+1)\hbar^2 = 6\hbar^2$ is one. For L_z we must first normalize the coefficients in front of the $Y_{l,m}$. The sum of the squares of the coefficients is $\frac{12\pi}{15}$ and hence we get the following expression: (Note this last step is not nescesary but convinient otherwise I would have to keep track of sum of squared coeficients)

$$\psi(\boldsymbol{r}) = Nr^2e^{-\alpha r}\sqrt{\frac{12\pi}{15}}\left(\frac{1}{4i}\sqrt{\frac{8}{3}}(Y_{2,2}-Y_{2,-2}) + \frac{i-1}{2}\sqrt{\frac{2}{3}}Y_{2,1} + \frac{1+i}{2}\sqrt{\frac{2}{3}}Y_{2,-1}\right)$$

The probability to get $m=2$ is $|\frac{1}{4}\sqrt{\frac{8}{3}}|^2 = \frac{1}{6}$, for $m=1$ is $|\frac{1-i}{2}\sqrt{\frac{2}{3}}|^2 = \frac{1}{3}$, for $m=0$ is $=0$ for $m=-1$ is $|\frac{1+i}{2}\sqrt{\frac{2}{3}}|^2 = \frac{1}{3}$, and for $m=-2$ is $|\frac{1}{4}\sqrt{\frac{8}{3}}|^2 = \frac{1}{6}$,

The expectation value $\langle L^2\rangle = 6\hbar^2$ and for $\langle L_z\rangle = 2\frac{1}{6} + 1\frac{1}{3} + 0 - 1\frac{1}{3} - 2\frac{1}{6} = 0\hbar$.

4. 2 particles A and B, 3 states with energy 0, ϵ and 2ϵ

a) Classical

state	0	ϵ	2ϵ	energy
1	AB	-	-	0
2	-	AB	-	2ϵ
3	-	-	AB	4ϵ
4	A	B	-	ϵ
5	B	A	-	ϵ
6	A	-	B	2ϵ
7	B	-	A	2ϵ
8	-	A	B	3ϵ
9	-	B	A	3ϵ

and $Z = 1 + 2e^{-\epsilon/\tau} + 3e^{-2\epsilon/\tau} + 2e^{-3\epsilon/\tau} + e^{-4\epsilon/\tau}$

b) Bosons

state	0	ϵ	2ϵ	energy
1	AA	-	-	0
2	-	AA	-	2ϵ
3	-	-	AA	4ϵ
4	A	A	-	ϵ
6	A	-	A	2ϵ
8	-	A	A	3ϵ

and $Z = 1 + e^{-\epsilon/\tau} + 2e^{-2\epsilon/\tau} + e^{-3\epsilon/\tau} + e^{-4\epsilon/\tau}$

c) Fermions

state	0	ϵ	2ϵ	energy
4	A	A	-	ϵ
6	A	-	A	2ϵ
8	-	A	A	3ϵ

and $Z = e^{-\epsilon/\tau} + e^{-2\epsilon/\tau} + e^{-3\epsilon/\tau}$

5. $Z = (Z_1)^N$ eftersom de är oberoende och särskiljbara. $Z_1 = \sum_{n=0,2,4,\ldots} e^{-\left(n+\frac{1}{2}\right)\hbar\omega/\tau} =$ [bryt ut $e^{-\hbar\omega/2\tau}$, geometrisk serie med faktorn $e^{-2\hbar\omega/\tau}$] $= e^{\hbar\omega/2\tau}\cdot\frac{1}{1-e^{2\hbar\omega/\tau}}$. Tillståndssumman ges av $Z = \left(e^{-\hbar\omega/2\tau}\cdot\frac{1}{1-e^{-2\hbar\omega/\tau}}\right)^N$. Fria energin ges i sin tur av

$F = -N\tau \ln\left(e^{-\hbar\omega/2\tau} \cdot \frac{1}{1-e^{-2\hbar\omega/\tau}}\right) = \frac{N\hbar\omega}{2} + N\tau \ln\left(1 - e^{-2\hbar\omega/\tau}\right)$. Entropin ges av $\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_V = -N \ln\left(1 - e^{-2\hbar\omega/\tau}\right) + \frac{2N\hbar\omega}{\tau} \frac{e^{-2\hbar\omega/\tau}}{1-e^{-2\hbar\omega/\tau}}$ och således ges värmekapacitiviteten av $C_v = \tau\left(\frac{\partial \sigma}{\partial \tau}\right)_V = \frac{2N\hbar\omega}{\tau^2}\frac{1}{e^{2\hbar\omega/\tau}-1} - \frac{2N\hbar\omega}{\tau^2}\frac{1}{e^{2\hbar\omega/\tau}-1} + \frac{4N\hbar^2\omega^2}{\tau^2}\frac{e^{2\hbar\omega/\tau}}{\left(e^{2\hbar\omega/\tau}-1\right)^2} = \frac{4N\hbar^2\omega^2}{\tau^2}\frac{e^{2\hbar\omega/\tau}}{\left(e^{2\hbar\omega/\tau}-1\right)^2}$

Systemet blir mer kvantmekanisk med urvalsregeln ty energin för lägsta excitationen blir dubbelt så hög och därmed krävs också dubbelt så hög temperatur (se Boltzmann faktorn) för att excitera detta tillstånd i motsvarande grad som för systemet utan restriktioner.