LULEÅ UNIVERSITY OF TECHNOLOGY
Division of Physics

Solution to written exam in Quantum Physics and Statistical Physics MTF131

Examination date: 2006-08-26

1. The line that is special (due to intensity) is $\lambda = 470.22$nm with intensity 200. The Helium ion has $Z = 2$ and hence energys $E_n = -\frac{54.24}{n^2}$eV. Try to find a start of the series. The energy of $\lambda = 658.30$nm is $E = h\nu = \frac{hc}{\lambda} = \frac{6.626\cdot 10^{-34}\cdot 2.9979\cdot 10^{8}}{6.5830\cdot 10^{-7}\cdot 1.6022\cdot 10^{-19}} = 1.8833$eV A similar calculation gives for the other lines in the series: 2.28306, 2.54250, 2.72037, 2.84760, 2.94174, 3.01333, 3.06905 and for the special line 2.63667eV

   As Balmer series in Hydrogen is for transitions down to level n=2 we have to go higher up for the Helium ion. If we try n=4 we have transitions from m=5, 6, 7, etc. The corresponding energys will be: $54.24(\frac{1}{4^2} - \frac{1}{5^2})$=1.22eV, the next one will be: $54.24(\frac{1}{4^2} - \frac{1}{6^2})$=1.8833eV, $54.24(\frac{1}{4^2} - \frac{1}{7^2})$=2.28306V and so on. So these are down to n=4 from level m=6, 7, 8, 9, 10, 11, 12 and 13. The special line a similar analysis gives from m=4 to n=3.

2. 2 particles A and B, 3 states with energy 0, ϵ and 3ϵ **a**) Classical

state	0	ϵ	3ϵ	energy
1	AB	-	-	0
2	-	AB	-	2ϵ
3	-	-	AB	6ϵ
4	A	B	-	ϵ
5	B	A	-	ϵ
6	A	-	B	3ϵ
7	B	-	A	3ϵ
8	-	A	B	4ϵ
9	-	B	A	4ϵ

and $Z = 1 + 2e^{-\epsilon/\tau} + e^{-2\epsilon/\tau} + 2e^{-3\epsilon/\tau} + 2e^{-4\epsilon/\tau} + e^{-6\epsilon/\tau}$

b) Bosons

state	0	ϵ	3ϵ	energy
1	AA	-	-	0
2	-	AA	-	2ϵ
3	-	-	AA	6ϵ
4	A	A	-	ϵ
6	A	-	A	3ϵ
8	-	A	A	4ϵ

and $Z = 1 + e^{-\epsilon/\tau} + e^{-2\epsilon/\tau} + e^{-3\epsilon/\tau} + e^{-4\epsilon/\tau} + e^{-6\epsilon/\tau}$

c) Fermions

state	0	ϵ	3ϵ	energy
4	A	A	-	ϵ
6	A	-	A	3ϵ
8	-	A	A	4ϵ

and $Z = e^{-\epsilon/\tau} + e^{-3\epsilon/\tau} + e^{-4\epsilon/\tau}$

3. $Z = 1 + e^{\frac{mB}{\tau}} + e^{-\frac{mB}{\tau}} \approx 1 + 1 + \frac{mB}{\tau} + \frac{1}{2}\left(\frac{mB}{\tau}\right)^2 + 1 - \frac{mB}{\tau} + \frac{1}{2}\left(\frac{mB}{\tau}\right)^2 = 3(1 + \frac{1}{3}\left(\frac{mB}{\tau}\right)^2)$

   $F = -\tau \ln Z = -\tau\left[\ln 3 + \ln(1 + \frac{1}{3}\left(\frac{mB}{\tau}\right)^2)\right] \approx -\tau\left[\ln 3 + \frac{1}{3}\left(\frac{mB}{\tau}\right)^2)\right]$

   $\sigma = -\frac{\partial F}{\partial \tau}_V = \ln 3 - \frac{1}{3}\left(\frac{mB}{\tau}\right)^2)$. The decrease in entropy is $\frac{1}{3}\left(\frac{mB}{\tau}\right)^2)$ and $A = \frac{1}{3}(mB)^2$

4. The eigenfunctions of the infinite square well in one dimension are (Here a solution of the S.E. in one dimesion is adequate)

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \quad \text{and the eigenenergys are } E_n = \frac{n^2\pi^2\hbar^2}{2Ma^2} \quad \text{where} \quad n = 1, 2, 3, ...$$

$$\psi_m(x) = \sqrt{\frac{2}{\sqrt{2}a}} \sin \frac{m\pi y}{\sqrt{2}a} \quad \text{and the eigenenergys are } E_n = \frac{n^2\pi^2\hbar^2}{2M2a^2} \quad \text{where} \quad m = 1, 2, 3, ...$$

In two dimensions the eigenfunctions and eigenenergys for the rectangular well are (Here an argument about separation of variables is needed)

$\Psi_{n,m}(x, y) = \psi_n(x) \cdot \psi_m(y)$ and eigenenergys $E_{n,m} = E_n + E_m$ where $n = 1, 2, 3, ..$ and $m = 1, 2, 3, ..$

a) The eigenfunctions inside the rectangle

$$\Psi_{n,m}(x, y) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \cdot \sqrt{\frac{2}{\sqrt{2}a}} \sin \frac{m\pi y}{\sqrt{2}a} \quad \text{where} \quad n = 1, 2, 3, .. \text{ and } m = 1, 2, 3, ..$$

The eigenfunctions outside the rectangle $\Psi_{n,m}(x, y) = 0$

b) The five lowest eigenenergies are

$$E_{n,m} = \frac{\pi^2\hbar^2}{2Ma^2}(n^2 + \frac{m^2}{2}), \quad \text{where the 5 lowest are} \quad (n^2 + \frac{m^2}{2}) = 1.5, 3, 4.5, 5.5, 6, \ \mathit{8.5} \text{ and } \mathit{9}.$$

c) The five lowest eigenenergys have degeneracys (not degenerate !) as follows:

$$E_{1,1} = \text{ one state } (n^2 + \frac{m^2}{2} = 1.5)$$

$$E_{1,2} = \text{ one state } (n^2 + \frac{m^2}{2} = 3)$$

$$E_{2,1} = \text{ one state } (n^2 + \frac{m^2}{2} = 4.5)$$

$$E_{1,3} = \text{ one state } (n^2 + \frac{m^2}{2} = 5.5)$$

$$E_{2,2} = \text{ one state } (n^2 + \frac{m^2}{2} = 6)$$

$$E_{2,3} = \text{ one state } (n^2 + \frac{m^2}{2} = 8.5)$$

$$E_{1,4} = \text{ one state } (n^2 + \frac{m^2}{2} = 9)$$

5. A measurement of the spin component in the direction $\hat{n} = \cos\varphi\hat{x} + \sin\varphi\hat{y}$ gives the value $\hbar/2$. The spin operator $S_{\hat{n}}$ is

$$S_{\hat{n}} = \frac{\hbar}{2}\begin{pmatrix} 0 & \cos\varphi - i\sin\varphi \\ \cos\varphi - i\sin\varphi & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & e^{-i\varphi} \\ e^{i\varphi} & 0 \end{pmatrix}$$

The eigenvalue equation is

$$S_{\hat{n}}\chi = \lambda\chi \Leftrightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & e^{-i\varphi} \\ e^{i\varphi} & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix} \tag{1}$$

We find the eigenvalues from

$$\begin{vmatrix} -\lambda & \frac{\hbar}{2}e^{-i\varphi} \\ \frac{\hbar}{2}e^{i\varphi} & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda = \pm\frac{\hbar}{2}$$

(a) The spin state corresponding to $\lambda = +\hbar/2$ must satisfy the eigenvalue equation Eq. (1), *i.e.*

$$\chi_{\hat{n}+} = \begin{pmatrix} a \\ b \end{pmatrix} = b\begin{pmatrix} e^{-i\varphi} \\ 1 \end{pmatrix} \Rightarrow \chi_{\hat{n}+} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\varphi} \\ 1 \end{pmatrix},$$

where the normalization condition $|a|^2 + |b|^2 = 1$ was used in the last step. Other correct solutions can be found by a multiplication with an arbitrary phase factor $\exp(i\alpha)$.

(b) A general spin state can be written as $\chi = a\chi_+ + b\chi_-$, where χ_+ is spin up and χ_- is spin down in z-direction. For $\chi_{\hat{n}+}$ we find that the probability to measure spin up, *i.e.* $S_z = \hbar/2$ is $|a|^2 = |e^{-i\varphi}/\sqrt{2}|^2 = 1/2$, and that the probability to measure spin down, *i.e.* $S_z = -\hbar/2$ is $|b|^2 = |1/\sqrt{2}|^2 = 1/2$.