LULEÅ UNIVERSITY OF TECHNOLOGY
Division of Physics

Solution to written exam in QUANTUM PHYSICS AND STATISTICAL PHYSICS F0018T / MTF131
Examination date: 2007-12-21
Note solutions are more detailed compared to previous solutions, earlier than October 2007.

1. The general relation for the specific heat is $C_v = \tau \left(\frac{\partial\sigma}{\partial\tau}\right)_v$

   a: in case of the conduction electrons we have $C_v = \gamma\tau$ these two relations combine to give $\gamma\tau = \tau\left(\frac{\partial\sigma}{\partial\tau}\right)_v$ leading to $\frac{\partial\sigma}{\partial\tau} = \gamma = constant$. and hence integrating to $\sigma \propto \tau +$'new constant' where the 'new constant' is zero as the the entropy is zero at temperature absolute zero. If the temperature increases from $\tau = 100$K to 400 K the entropy σ will increase by a factor **4**.

   b: In the case of the electro magnetic field we have that the energy density is $u \propto \tau^4$ (Stefan–Boltzmann T^4 law) and hence we have for the specific heat $C_v \propto \tau^3$ (Note the similarity to phonons at low temperature the Debye T^3 law). As in a) we arrive at $\frac{\partial\sigma}{\partial\tau} \propto \tau^2$ and hence $\sigma \propto \tau^3$. If the temperature is raised from 500K to 1500K the entropy σ will increase by a factor of $(\frac{1500}{500})^3 = 27$ that is a factor of **27**.

2. (a) $i\hbar\frac{\partial^2}{\partial t^2}\sin\omega t = i\hbar\omega\frac{\partial}{\partial t}\cos\omega t = -i\hbar\omega^2\sin\omega t$ **YES**

   (b) $-i\hbar\frac{\partial}{\partial z}C(1+z^2) = -i\hbar C(0+2z)$ **NO**

   (c) $-i\hbar\frac{\partial^2}{\partial z^2}(C_1e^{ikz} + C_2e^{-ikz}) = -i\hbar ik\frac{\partial}{\partial z}(C_1e^{ikz} - C_2e^{-ikz}) = -i\hbar k^2(C_1e^{ikz} + C_2e^{-ikz})$ **YES**

   (d) $-\frac{\hbar}{2}\frac{\partial}{\partial z}Ce^{-3z} = -\frac{\hbar}{2}C(-3)e^{-3z} \;\propto\; \psi(z)$ **YES**

   (e) $\frac{C}{2}(z^2 - \frac{\partial^2}{\partial z^2})ze^{-\frac{1}{2}z^2} =?$ This has to be done in some steps. Start by doing this derivative first: $-\frac{\partial^2}{\partial z^2}ze^{-\frac{1}{2}z^2} = -\frac{\partial}{\partial z}(e^{-\frac{1}{2}z^2} - z^2e^{-\frac{1}{2}z^2}) = -(-ze^{-\frac{1}{2}z^2} - 2ze^{-\frac{1}{2}z^2} + z^3e^{-\frac{1}{2}z^2}) = 3ze^{-\frac{1}{2}z^2} - z^3e^{-\frac{1}{2}z^2}$.
   Now you go back to the start: $\frac{C}{2}(z^2 - \frac{\partial^2}{\partial z^2})ze^{-\frac{1}{2}z^2} = \frac{C}{2}(z^3e^{-\frac{1}{2}z^2} + 3ze^{-\frac{1}{2}z^2} - z^3e^{-\frac{1}{2}z^2}) = \frac{C}{2}(+3ze^{-\frac{1}{2}z^2}) = \propto\; \psi(z)$ **YES**

   (f) $\frac{C}{2}(z^2 - \frac{\partial^2}{\partial z^2})e^{-\frac{1}{2}z^2} = \frac{C}{2}(z^2e^{-\frac{1}{2}z^2} - \frac{\partial}{\partial z}(-ze^{-\frac{1}{2}z^2})) = \frac{C}{2}(z^2e^{-\frac{1}{2}z^2} - (-e^{-\frac{1}{2}z^2} + z^2e^{-\frac{1}{2}z^2})) = \frac{C}{2}e^{-\frac{1}{2}z^2} \;\propto\; \psi(z)$ **YES**

3. The eigenfunctions and eigenvalues of the free-particle Hamiltonian are found by solving the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2u(x)}{dx^2} + V(x)u(x) = Eu(x),$$

   with $V(x)$ zero everywhere. Thus, the eigenvalue equation reads

$$\frac{d^2u(x)}{dx^2} + k^2u(x) = 0,$$

   where $k^2 = 2mE/\hbar^2$. The eigenfunctions are given by the plane waves e^{ikx} and e^{-ikx}, or linear combinations of these, as *e.g.* $\sin kx$ and $\cos kx$.

   (a) The wave function of the particle at $t = 0$ is given by

$$\psi(x,0) = \sin^3 kx.$$

This is not an eigenfunction in itself but it can be written as using the Euler relations

$$\psi(x,0) = \left(\frac{e^{ikx} - e^{-ikx}}{2i}\right)^3 = -\frac{1}{8i}\left(e^{i3kx} - 3e^{ikx} + 3e^{-ikx} - e^{-i3kx}\right) = +\frac{3}{4}\sin kx - \frac{1}{4}\sin 3kx. \tag{1}$$

Thus, $\psi(x,0)$ can be written as a superposition the plane waves $\sin k_1 x$ and $\sin k_2 x$, with $k_1 = k$ and $k_2 = 3k$

(b) The energy of a plane wave e^{ikx} is given by $E = \hbar^2 k^2/2m$. Thus, the energy of $\sin k_1 x$ is $E_1 = \hbar^2 k^2/2m$ and the energy of $\sin k_2 x$ is $E_2 = \hbar^2 k_2^2/2m = 9\hbar^2 k^2/2m$.

(c) The function $u(x) = e^{ikx}$ is a solution to the the time-independent Schrödinger equation. The corresponding solutions to the time-dependent Schrödinger equation are given by $u(x)T(t)$,with $T(t) = e^{-iEt/\hbar}$. Therefore, $u(x)T(t) = e^{i(kx-Et/\hbar)}$. A sum of solutions of this form is also a solution, since the Schrödinger equation is linear. This means that if $\psi(x,0)$ is given by equation (1), then the time dependent solution is given by

$$\begin{aligned}\psi(x,t) &= -\frac{1}{8i}\left[\left(e^{i3kx} - e^{-i3kx}\right)e^{-iE_2 t/\hbar} + 3\left(-e^{ikx} + e^{-ikx}\right)e^{-iE_1 t/\hbar}\right] \\ &= \frac{3}{4i}\sin(kx)e^{-iE_1 t/\hbar} - \frac{1}{4i}\sin(3kx)e^{-iE_2 t/\hbar} \end{aligned} \tag{2}$$

where

$$E_1 = \frac{\hbar^2 k^2}{2m} \quad \text{and} \quad E_2 = \frac{9\hbar^2 k^2}{2m} \tag{3}$$

4. (a) Let the commutator act on a wave function $\Psi(y)$ and $p_y = -i\hbar\frac{d}{dy}$
$[y^2, p_y^2]\Psi(y) = -\hbar^2(y^2\frac{d^2\Psi(y)}{dy^2} - \frac{d^2(y^2\Psi(y))}{dy^2}) = -\hbar^2\left(y^2\frac{d^2\Psi(y)}{dy^2} - y^2\frac{d^2\Psi(y)}{dy^2} - 4y\frac{d\Psi(y)}{dy} - 2\Psi(y)\right) = +\hbar^2 2\Psi(y) + 4y\hbar^2\frac{d\Psi(y)}{dy} = \left(+\hbar^2 2 + i4\hbar y p_y\right)\Psi(y)$ concluding for the commutator:
$[y^2, p_y^2] = +2\hbar^2 + 4i\hbar y p_y = +2\hbar^2 + 4\hbar^2 y\frac{d}{dy}$.

(b) The energy levels for a hydrogen like system are given by: $E_n = -13.6\frac{Z^2}{n^2}$ [eV], here we have $Z = 4 :\ \Delta E = E(2s) - E(1s) = E_2 - E_1 = -13.54 \cdot (\frac{16}{2^2} - \frac{16}{1^2}) = 13.54 \cdot \frac{16\cdot 3}{4} = 162.48$ eV

(c) The angular part of the wave function can be written as a spherical harmonic:

$$3\cos^2\theta - 1 \propto Y_{20}$$

Which gives $l = 2$ och $m = 0$. The part depending on r $(r^2/a_\mu^2)e^{-r/3a_\mu}$ corresponding to the principal quantum number $n = 3$ och $l = 2$ consistent with Y_{20}.

5. The energies of the states are given by $\epsilon_j = (j + \frac{1}{2})\hbar\omega$. Note that we can treat the oscillators as independent from each other. To find the fraction of oscillators in a particular state is the same as to ask for the probability of a oscillator to be in that particular state. In order to find the probabilities (fractions) we need to calculate the partition function for the system consisting of a single oscillator $Z = \sum e^{-(j+\frac{1}{2})\hbar\omega/\tau} = e^{-\frac{\hbar\omega}{2\tau}}\sum_{n=0}^{\infty} e^{-n\hbar\omega/\tau} = e^{-\frac{\hbar\omega}{2\tau}}\frac{1}{1-e^{-\hbar\omega/\tau}}$. Note the partition function is a geometric sum. At the characteristic temperature given by $\tau_{ch} = \hbar\omega$ the partition function is $Z = e^{-\frac{1}{2}}\frac{1}{1-e^{-1}} = \frac{1}{e^{1/2}-e^{-1/2}}$.
The fraction of oscillators in the ground state ($j = 0$) is given by
$f_0 = \frac{e^{-\epsilon_0/\tau}}{Z} = e^{-\frac{1}{2}}(e^{1/2} - e^{-1/2}) = (1 - e^{-1}) = 0.632.$
The next states ($j = 1$) fraction is given by $f_1 = \frac{e^{-\epsilon_1/\tau}}{Z} = e^{-\frac{3}{2}}(e^{1/2} - e^{-1/2}) = (e^{-1} - e^{-2}) = 0.233.$
The next states ($j = 2$) fraction is given by
$f_2 = \frac{e^{-\epsilon_2/\tau}}{Z} = e^{-\frac{5}{2}}(e^{1/2} - e^{-1/2}) = (e^{-2} - e^{-3}) = 0.0855.$