LULEÅ UNIVERSITY OF TECHNOLOGY
Division of Physics

Solution to written exam in QUANTUM PHYSICS AND STATISTICAL PHYSICS F0018T / MTF131

Examination date: 2008-10-30
Note solutions are more detailed compared to previous solutions, earlier than October 2007.

1. (a) To show that $\psi(\xi) = \xi e^{\xi^2/2}$ solves the differential equation put it in! The first derivative and second derivatives are:

$$\frac{d\psi(\xi)}{d\xi} = e^{\xi^2/2} + \xi^2 e^{\xi^2/2} \quad \text{and} \quad \frac{d^2\psi(\xi)}{d\xi^2} = \xi e^{\xi^2/2} + 2\xi e^{\xi^2/2} + \xi^3 e^{\xi^2/2} = 3\xi e^{\xi^2/2} + \xi^3 e^{\xi^2/2}$$

Now evaluate the following:

$$\frac{d^2\psi(\xi)}{d\xi^2} + (\lambda - \ \xi^2)\psi(\xi) = (\lambda+3)\xi e^{\xi^2/2} + \xi^3 e^{\xi^2/2} - \xi^3 e^{\xi^2/2} = (\lambda+3)\xi e^{\xi^2/2}$$

If $\lambda = -3$ the desired result is reached.

(b) No, the suggested solutions diverges and cannot be normalized. Therefore it does not describe a particle.

2. (a) There are several ways to determine A. One is to integrate and use the normalization condition to solve for A. A different path (done here) is to write the given wave function in terms of eigenfunctions. The eigenfunctions are (PH) $\psi(x) = \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a})$. We can directly conclude that the given wave function consists of $n = 1$ and $n = 5$ functions, we can write:

$$\psi(x,0) = \frac{A\sqrt{2}}{\sqrt{2a}}\sin\left(\frac{\pi x}{a}\right) + \frac{\sqrt{2}}{\sqrt{2\cdot 5a}}\sin\left(\frac{5\pi x}{a}\right) = \frac{A}{\sqrt{2}}\psi_1(x,0) + \frac{1}{\sqrt{10}}\psi_5(x,0)$$

As both eigenfunctions are orthonormal the normalisation integral reduces to $\frac{A^2}{2} + \frac{1}{10} = 1$ and hence $A = \sqrt{\frac{18}{10}} = \sqrt{\frac{9}{5}} = \frac{3}{\sqrt{5}}$

(b) The wave function contains only $n = 1$ and $n = 5$ eigenfunctions and therefore the only possible outcome of an energy meassurement are $E_1 = \frac{\hbar^2\pi^2}{2ma^2}$ with probability $\frac{A^2}{2} = 0.9$ and $E_5 = \frac{\hbar^2\pi^2}{2ma^2}25$ with probability 1 - 0.9 = 0.1 . The average energy is given by $<E>= 0.9E_1 + 0.1E_5 = \frac{\hbar^2\pi^2}{2ma^2}(0.9 + 0.1\cdot 25) = 3.4\cdot\frac{\hbar^2\pi^2}{2ma^2} = 1.7\cdot\frac{\hbar^2\pi^2}{ma^2}$

(c) The time dependent solution is given by $\Psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-iE_n t/\hbar}$ and hence

$$\Psi(x,t) = \sqrt{\frac{9}{10}}\psi_1(x,0)e^{-i\frac{\hbar\pi^2 t}{2ma^2}} + \frac{1}{\sqrt{10}}\psi_5(x,0)e^{-i\frac{25\hbar\pi^2 t}{2ma^2}}$$

3. a) The partition sum is given by: $Z = \sum_{n_1=0,n_2=0}^{\infty} e^{-(n_1+n_2+1.0)\hbar\omega/k_BT} = \sum_{n=0}^{\infty} g(n)e^{-(n+1.0)\hbar\omega/k_BT}$, where $g(n)$ is the degeneracy of the energy levels and $n = n_1 + n_2$. There are two ways to evaluate this sum. One simple and one more elaborate. Only the simple solution is presented here. The sum for Z can be done as a product of two separate independent geometric sums.
$Z = \sum_{n_1=0,n_2=0}^{\infty} e^{-(n_1+n_2+1.0)\hbar\omega/k_BT} = \sum_{n_1=0}^{\infty} e^{-(n_1+0.5)\hbar\omega/k_BT}\cdot\sum_{n_2=0}^{\infty} e^{-(n_2+0.5)\hbar\omega/k_BT} = \left(\sum_{n=0}^{\infty} e^{-(n+0.5)\hbar\omega/k_BT}\right)^2 = e^{-\hbar\omega/k_BT}\left(\sum_{n=0}^{\infty} e^{-n\hbar\omega/k_BT}\right)^2 = e^{-\hbar\omega/k_BT}\left(\frac{1}{1-e^{-\hbar\omega/k_BT}}\right)^2 =$ and we arrive at the following for the partition function Z:

$$Z = \left(\frac{1}{e^{+\hbar\omega/2k_BT} - e^{-\hbar\omega/2k_BT}}\right)^2 = \text{ or } = e^{-\hbar\omega/k_BT}\left(\frac{1}{1-e^{-\hbar\omega/k_BT}}\right)^2.$$

b) There is one state of the lower energy and two states with the next higher energy. The probability to find the oscillator in a state of energy is proportional to the Boltzmann factor, we arrive at the following equation. $1e^{-1,0\hbar\omega/k_BT} = 2e^{-2,0\hbar\omega/k_BT}$ and $e^{1\hbar\omega/k_BT} = 2$ which evaluates to $T = \frac{1\hbar\omega}{k_B \ln 2}$ or if you prefer τ, $\tau = \frac{1\hbar\omega}{\ln 2}$, which is equally correct.

c) The partition sum at this specific temperature is given by: ($k_BT = \tau = \frac{1\hbar\omega}{\ln 2}$) we arrive at the following

$$Z = \left(\frac{1}{e^{+\frac{\ln 2}{2}} - e^{-\frac{\ln 2}{2}}}\right)^2 = \left(\frac{1}{\sqrt{2} - \frac{1}{\sqrt{2}}}\right)^2$$

We continue with the calculation of the probability: (we may choose any of the two energies as their probabilities will be equal at the temperature in question: ($1e^{-1,0\hbar\omega/k_BT} = 2e^{-2,0\hbar\omega/k_BT}$)

$$P = e^{-\hbar\omega/k_BT} / \left(\frac{1}{\sqrt{2} - \frac{1}{\sqrt{2}}}\right)^2 = 2^{-1}\left(\sqrt{2} - \frac{1}{\sqrt{2}}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2\left(\sqrt{2} - \frac{1}{\sqrt{2}}\right)^2 =$$

$$\left(1 - \frac{1}{2}\right)^2 = \frac{1}{4} = 0.25$$

The probability to be in a state of one of these energys is $P = \frac{8}{27} = 0.25$

4. At the beginning the entropy in part 1 and 2 is given by $\sigma_{1,2} = N_{1,2}\left[\ln\left(\frac{n_Q}{n_{1,2}}\right) + \frac{5}{2}\right]$ and after mixing it is given by (same temperature): $\sigma_e = N_e\left[\ln\left(\frac{n_Q}{n_e}\right) + \frac{5}{2}\right]$.

The change of entropy is (increase):
$\Delta\sigma = \sigma_e - \sigma_1 - \sigma_2 = 2N\left[\ln\left(\frac{n_Q 3V}{2N}\right) + \frac{5}{2}\right] - N\left[\ln\left(\frac{n_Q 2V}{N}\right) + \frac{5}{2}\right] - N\left[\ln\left(\frac{n_Q V}{N}\right) + \frac{5}{2}\right] =$
$2N\ln\left(\frac{n_Q 3V}{2N}\right) - N\ln\left(\frac{n_Q 2V}{N}\right) - N\ln\left(\frac{n_Q V}{N}\right) =$
$2N\ln n_Q - N\ln n_Q - N\ln n_Q + 2N\ln 3V - N\ln 2V - N\ln V - 2N\ln 2N + N\ln N + N\ln N =$
$N\left(\ln(9V^2\frac{1}{2V}\frac{1}{V}) - \ln(4N^2N^{-1}N^{-1})\right) = N\ln(3^2 2^{-3}V^0N^0) = N\ln(\frac{9}{8}) = Nk_B\ln(\frac{9}{8})$

5. **a**) The eigenfunctions of the infinite square well in one dimension are (Here a solution of the S.E. in one dimesion is adequate)

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} \quad \text{and the eigenenergys are } E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \quad \text{where} \quad n = 1, 2, 3, ...$$

The five lowest one particle energies are: $\frac{1^2\pi^2\hbar^2}{2mL^2}$, $\frac{2^2\pi^2\hbar^2}{2mL^2}$, $\frac{3^2\pi^2\hbar^2}{2mL^2}$, $\frac{4^2\pi^2\hbar^2}{2mL^2}$ and $\frac{5^2\pi^2\hbar^2}{2mL^2}$.

b) The many particle Hamiltonian commutes with the particle index exchange operator. Therefore also the solutions are also eigenfunctions to this operator. They can have eigenvalue +1 for bosons and -1 for fermions.

The eigenfunctions inside the well for two noninteracting and integer spin particles (=bosons). Start the construction with the products:

$$\psi_{n,m}(x_1, x_2) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x_1}{L} \cdot \sqrt{\frac{2}{L}}\sin\frac{m\pi x_2}{L} \quad \text{where} \quad n = 1, 2, 3, .. \text{ and } m = 1, 2, 3, ..$$

where index n and x_1 is for one particle and m and x_2 for the other. The above function is not en eigenfunction of the particle index exchange operator but the following combination is.

$$\Psi_{n,m}(x_1, x_2) = \frac{1}{\sqrt{2}}(\psi_{n,m}(x_1, x_2) + \psi_{n,m}(x_2, x_1))$$

c) The fermion case. The eigenvalue is -1. The construction is similar but there is a sign change.

$$\Psi_{n,m}(x_1, x_2) = \frac{1}{\sqrt{2}}(\psi_{n,m}(x_1, x_2) - \psi_{n,m}(x_2, x_1))$$