

## L27 Omvänd substitution (Adams 6.3)

①

Tridigare: Substitution  $u=g(x)$

Nu: Omvänd substitution  $x=f(u)$

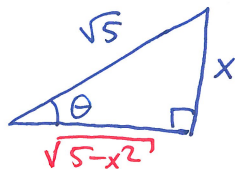
För integraler som innehåller  $\sqrt{a^2-x^2}$  för något  $a>0$

Så kan ett variabelbyte  $x=a \sin(\theta)$ , där det är underförstått att  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ , så att  $\sin(\theta)$  är inverterbar,  $\cos(\theta) \geq 0$  och

$$\sqrt{a^2-x^2} = \sqrt{a^2(1-\sin^2(\theta))} = \sqrt{a^2 \cos^2(\theta)} = a \cos(\theta)$$

$$\begin{aligned} \text{Ex 1 } \int \frac{1}{(5-x^2)^{3/2}} dx &= \left[ \begin{array}{l} x = \sqrt{5} \sin(\theta) \\ dx = \sqrt{5} \cos(\theta) d\theta \end{array} \right] = \int \frac{\cos(\theta)}{(5(1-\sin^2(\theta)))^{3/2}} d\theta \\ &= \int \frac{\sqrt{5} \cos(\theta)}{5^{3/2} \cos^3(\theta)} d\theta = \frac{1}{5} \int \frac{1}{\cos^2(\theta)} d\theta = \\ &= \frac{1}{5} \tan(\theta) + C \end{aligned}$$

Hur skriva detta som funktion av  $x$ ?



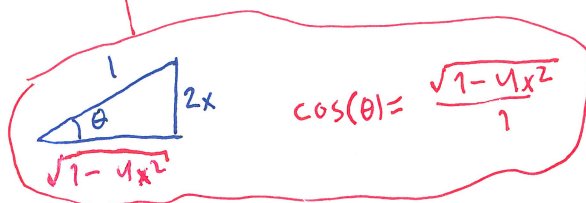
$$\tan(\theta) = \frac{x}{\sqrt{5-x^2}}$$

Alltså är

$$\int \frac{1}{(5-x^2)^{3/2}} dx = \frac{x}{5\sqrt{5-x^2}} + C$$

6.3.2

$$\begin{aligned} \int \frac{x^2}{\sqrt{1-4x^2}} dx &= \left[ \begin{array}{l} 2x = \sin(\theta) \\ x = \frac{1}{2} \sin(\theta) \\ dx = \frac{1}{2} \cos(\theta) d\theta \end{array} \right] = \int \frac{\frac{1}{4} \sin^2(\theta)}{\cos(\theta)} \cdot \frac{1}{2} \cos(\theta) d\theta = \\ &= \frac{1}{8} \int \frac{1 - \cos(2\theta)}{2} d\theta = \frac{1}{16} \left( \theta - \frac{1}{2} \sin(2\theta) \right) + C = \frac{1}{16} \left( \sin^{-1}(2x) - \frac{1}{2} \cdot 2 \sin(\theta) \cos(\theta) \right) = \\ &= \frac{1}{16} \left( \sin^{-1}(2x) - 2x \sqrt{1-4x^2} \right) + C \end{aligned}$$



② För integraler som innehåller  $\sqrt{a^2+x^2}$  eller  $\frac{1}{x^2+a^2}$  med  $a > 0$

kan det ofta bli enklare med variabelbyte  $x = a \tan(\theta)$ .

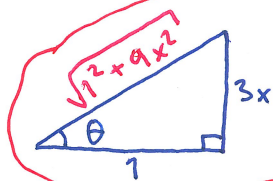
Underförstått med  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ , där  $\tan(\theta)$  är inverterbar.

Viktigt för att kunna sätta  $\theta = \tan^{-1}(x/a)$  om det behövs efter integrering. se tex Ex 3(b) nedan.

Da är  $\cos(\theta) \geq 0$  och  $\sqrt{a^2+x^2} = a\sqrt{1+\tan^2(\theta)} = a\sqrt{\frac{1}{\cos^2(\theta)}} = \frac{a}{\cos(\theta)}$ .

Ex 3 (b)

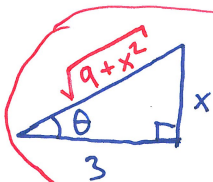
$$\begin{aligned} \int \frac{1}{(1+9x^2)^2} dx &= \int \frac{1}{(1+(3x)^2)^2} dx = \left[ \begin{array}{l} 3x = \tan(\theta) \\ x = \frac{1}{3} \tan(\theta) \\ dx = \frac{1}{3\cos^2(\theta)} d\theta \end{array} \right] = \\ &= \int \frac{1}{(1+\tan^2(\theta))^2 3\cos^2(\theta)} d\theta = \int \frac{\cos^4(\theta)}{3\cos^2(\theta)} d\theta = \frac{1}{3} \int \cos^2(\theta) d\theta = \\ &= \frac{1}{3} \int \frac{1+\cos(2\theta)}{2} d\theta = \frac{1}{3} \left( \frac{\theta}{2} + \frac{1}{2} \cdot \frac{1}{2} \sin(2\theta) \right) + C = \\ &= \frac{1}{3} \left( \frac{\theta}{2} + \frac{1}{4} \cdot 2\sin(\theta)\cos(\theta) \right) + C = \underline{\underline{\frac{1}{6} \left( \tan^{-1}(3x) + \frac{3x}{1+9x^2} \right) + C}} \end{aligned}$$



$$\begin{aligned} \sin(\theta) &= \frac{3x}{\sqrt{1+9x^2}} \\ \cos(\theta) &= \frac{1}{\sqrt{1+9x^2}} \end{aligned}$$

6.3.10

$$\begin{aligned} \int \frac{\sqrt{9+x^2}}{x^4} dx &= \left[ \begin{array}{l} x = 3\tan(\theta) \\ dx = \frac{3}{\cos^2(\theta)} d\theta \end{array} \right] = \int \frac{3/\cos(\theta)}{3^4 \tan^4(\theta)} \cdot \frac{3}{\cos^2(\theta)} d\theta = \frac{1}{9} \int \frac{1}{\tan^4(\theta)\cos^3(\theta)} d\theta \\ &= \frac{1}{9} \int \frac{\cos(\theta)}{\sin^4(\theta)} d\theta = \left[ \begin{array}{l} u = \sin(\theta) \\ du = \cos(\theta) d\theta \end{array} \right] = \frac{1}{9} \int u^{-4} du = \frac{1}{9} \frac{u^{-3}}{-3} + C = \\ &= -\frac{1}{27 \sin^3(\theta)} = \underline{\underline{-\frac{(9+x^2)^{3/2}}{27x^3} + C}} \end{aligned}$$



$$\sin(\theta) = \frac{x}{\sqrt{9+x^2}}$$

För integraler där integranden är en kvot med polynom av  $\sin(\theta)$  och  $\cos(\theta)$  kan ett variabelbyte  $x = \tan(\frac{\theta}{2})$  skriva om detta till en rationell funktion med polynom av  $x$  i täljare och nämnare. Variabelbytet ger

$$\cos^2(\frac{\theta}{2}) = \frac{1}{\frac{1}{\cos^2(\frac{\theta}{2})}} = \frac{1}{1 + \tan^2(\frac{\theta}{2})} = \frac{1}{1+x^2}$$

$$\cos(\theta) = \cos(2 \cdot \frac{\theta}{2}) = 2\cos^2(\frac{\theta}{2}) - 1 = \frac{2}{1+x^2} - 1 = \frac{1-x^2}{1+x^2}$$

$$\sin(\theta) = 2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2}) = 2\tan(\frac{\theta}{2})\cos^2(\frac{\theta}{2}) = \frac{2x}{1+x^2}$$

$$dx = \frac{1}{2\cos^2(\frac{\theta}{2})} d\theta \Rightarrow d\theta = 2\cos^2(\frac{\theta}{2}) dx = \frac{2}{1+x^2} dx$$

Ex II

$$\int \frac{1}{2+\cos(\theta)} d\theta = \left[ \begin{array}{l} x = \tan(\frac{\theta}{2}) \\ \cos(\theta) = \frac{1-x^2}{1+x^2} \\ d\theta = \frac{2}{1+x^2} dx \end{array} \right] = \int \frac{1}{2 + \frac{1-x^2}{1+x^2}} \cdot \frac{2}{1+x^2} dx =$$

$$= \int \frac{2}{2+2x^2+1-x^2} dx = 2 \int \frac{1}{x^2+3} dx = \frac{2}{3} \int \frac{1}{(\frac{x}{\sqrt{3}})^2+1} dx$$

$$= \frac{2}{3} \sqrt{3} \tan^{-1}(\frac{x}{\sqrt{3}}) + C = \frac{2}{\sqrt{3}} \tan^{-1}(\frac{1}{\sqrt{3}} \tan(\frac{\theta}{2})) + C$$

Ex:

$$\int \frac{3}{6+2\sin(\theta)} d\theta = \left[ \begin{array}{l} x = \tan(\frac{\theta}{2}) \\ \sin(\theta) = \frac{2x}{1+x^2} \\ d\theta = \frac{2}{1+x^2} dx \end{array} \right] = \int \frac{3}{6 + \frac{4x}{1+x^2}} \cdot \frac{2}{1+x^2} dx =$$

$$= \int \frac{6}{6+6x^2+4x} dx = \int \frac{1}{x^2 + \frac{2}{3}x + 1} dx = \int \frac{1}{x^2 + 2 \cdot \frac{1}{3}x + \frac{1}{9} - \frac{1}{9} + 1} dx$$

$$= \int \frac{1 \cdot \frac{9}{8}}{((x + \frac{1}{3})^2 + \frac{8}{9}) \cdot \frac{9}{8}} dx = \frac{9}{8} \int \frac{1 \cdot \frac{\sqrt{8}}{3}}{(\frac{3}{\sqrt{8}}(x + \frac{1}{3}))^2 + 1} \cdot \frac{3}{\sqrt{8}} dx = \left[ u = \frac{3}{\sqrt{8}}(x + \frac{1}{3}) \right] =$$

$$= \frac{9}{8} \cdot \frac{\sqrt{8}}{3} \int \frac{1}{u^2+1} du = \frac{3}{\sqrt{8}} \tan^{-1}(u) + C = \frac{3}{\sqrt{8}} \tan^{-1}(\frac{3}{\sqrt{8}}(x + \frac{1}{3})) + C =$$

$$= \frac{3}{\sqrt{8}} \tan^{-1}(\frac{3}{\sqrt{8}}(\tan(\frac{\theta}{2}) + \frac{1}{3})) + C$$

④

För integraler som innehåller  $\sqrt[n]{ax+b}$  kan ett variabelbyte  $ax+b = u^n$  förenkla.

Ex 8

$$\int \frac{1}{1+\sqrt{2x}} dx = \left[ \begin{array}{l} 2x = u^2 \\ x = \frac{1}{2}u^2 \\ dx = u du \end{array} \right] = \int \frac{u}{1+u} du = \int \frac{1+u-1}{1+u} du =$$
$$= \int \left( 1 - \frac{1}{1+u} \right) du = u - \ln(1+u) + C = \underline{\underline{\sqrt{2x} - \ln(1+\sqrt{2x}) + C}}$$

Variabelbyte för att  
eliminera både  $\sqrt[3]{\phantom{x}}$   
och  $\sqrt{\phantom{x}}$

Ex 10

$$\int \frac{1}{x^{1/2}(1+x^{1/3})} dx = \left[ \begin{array}{l} x = u^6 \\ dx = 6u^5 \end{array} \right] = \int \frac{6u^5}{u^3(1+u^2)} dx =$$
$$= 6 \int \frac{1+u^2-1}{1+u^2} du = 6 \int \left( 1 - \frac{1}{1+u^2} \right) du = 6(u - \tan^{-1}(u)) + C =$$
$$= \underline{\underline{6(x^{1/6} - \tan^{-1}(x^{-1/6})) + C}}$$