L29 Omvänd substitution (Adams 6.3)

Tidigare: Substitution $u = g(x)$

Nu: Omvänd substitution $x = f(u)$

För integraler som innehåller $\sqrt{a^2 - x^2}$ för något $a > 0$ så kan ett variabelbyte $x = a\sin(\theta)$, där det är underförstått att $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, så att $\sin(\theta)$ är inverterbar, $\cos(\theta) \geq 0$ och

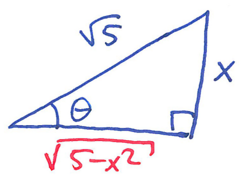
$$\sqrt{a^2 - x^2} = \sqrt{a^2(1 - \sin^2(\theta))} = \sqrt{a^2\cos^2(\theta)} = a\cos(\theta)$$

Ex 1
$$\int \frac{1}{(5-x^2)^{3/2}}\,dx = \begin{bmatrix} x = \sqrt{5}\sin(\theta) \\ dx = \sqrt{5}\cos(\theta)\,d\theta \end{bmatrix} = \int \frac{\cos(\theta)}{(5(1-\sin^2(\theta)))^{3/2}}\,d\theta$$

$$= \int \frac{\sqrt{5}\cos(\theta)}{5^{3/2}\cos^3(\theta)}\,d\theta = \frac{1}{5}\int \frac{1}{\cos^2(\theta)}\,d\theta =$$

$$= \frac{1}{5}\tan(\theta) + C$$

Hur skriva detta som funktion av x?



$\tan(\theta) = \frac{x}{\sqrt{5-x^2}}$

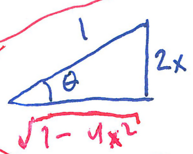
Alltså är

$$\int \frac{1}{(5-x^2)^{3/2}}\,dx = \frac{x}{5\sqrt{5-x^2}} + C$$

6.3.2
$$\int \frac{x^2}{\sqrt{1-4x^2}}\,dx = \begin{bmatrix} 2x = \sin(\theta) \\ x = \frac{1}{2}\sin(\theta) \\ dx = \frac{1}{2}\cos(\theta)\,d\theta \end{bmatrix} = \int \frac{\frac{1}{4}\sin^2(\theta)}{\cancel{\cos(\theta)}} \cdot \frac{1}{2}\cancel{\cos(\theta)}\,d\theta =$$

$$= \frac{1}{8}\int \frac{1 - \cos(2\theta)}{2}\,d\theta = \frac{1}{16}\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C = \frac{1}{16}\left(\sin^{-1}(2x) - \cancel{\frac{1}{2}}\cdot\cancel{2}\sin(\theta)\cos(\theta)\right) =$$

$$= \frac{1}{16}\left(\sin^{-1}(2x) - 2x\sqrt{1-4x^2}\right) + C$$



$\cos(\theta) = \frac{\sqrt{1-4x^2}}{1}$

② För integraler som innehåller $\sqrt{a^2+x^2}$ eller $\dfrac{1}{x^2+a^2}$ med $a>0$ kan det ofta bli enklare med variabelbyte $x = a\tan(\theta)$. Underförstått med $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, där $\tan(\theta)$ är inverterbar.

Viktigt för att kunna sätta $\theta = \tan^{-1}(x/a)$ om det behövs efter integrering. se tex Ex 3(b) nedan.

Då är $\cos(\theta) \geq 0$ och $\sqrt{a^2+x^2} = a\sqrt{1+\tan^2(\theta)} = a\sqrt{\dfrac{1}{\cos^2(\theta)}} = \dfrac{a}{\cos(\theta)}$.

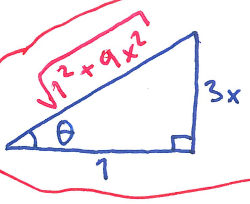
Ex 3 (b)

$$\int \frac{1}{(1+9x^2)^2}\,dx = \int \frac{1}{(1+(3x)^2)^2}\,dx = \left[\begin{array}{l} 3x = \tan(\theta) \\ x = \frac{1}{3}\tan(\theta) \\ dx = \frac{1}{3\cos^2(\theta)}\,d\theta \end{array}\right] =$$

$$= \int \frac{1}{(1+\tan^2(\theta))^2\, 3\cos^2(\theta)}\,d\theta = \int \frac{\cos^4(\theta)}{3\cos^2(\theta)}\,d\theta = \frac{1}{3}\int \cos^2(\theta)\,d\theta =$$

$$= \frac{1}{3}\int \frac{1+\cos(2\theta)}{2}\,d\theta = \frac{1}{3}\left(\frac{\theta}{2} + \frac{1}{2}\cdot\frac{1}{2}\sin(2\theta)\right) + C =$$

$$= \frac{1}{3}\left(\frac{\theta}{2} + \frac{1}{4}\cdot 2\sin(\theta)\cos(\theta)\right) + C = \frac{1}{6}\left(\tan^{-1}(3x) + \frac{3x}{1+9x^2}\right) + C$$



$\sin(\theta) = \dfrac{3x}{\sqrt{1+9x^2}}$

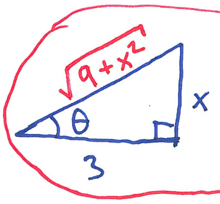
$\cos(\theta) = \dfrac{1}{\sqrt{1+9x^2}}$

6.3.10

$$\int \frac{\sqrt{9+x^2}}{x^4}\,dx = \left[\begin{array}{l} x = 3\tan(\theta) \\ dx = \frac{3}{\cos^2(\theta)}\,d\theta \end{array}\right] = \int \frac{3/\cos(\theta)}{3^4\tan^4(\theta)}\cdot\frac{3}{\cos^2(\theta)}\,d\theta = \frac{1}{9}\int \frac{1}{\tan^4(\theta)\cos^3(\theta)}\,d\theta$$

$$= \frac{1}{9}\int \frac{\cos(\theta)}{\sin^4(\theta)}\,d\theta = \left[\begin{array}{l} u = \sin(\theta) \\ du = \cos(\theta)\,d\theta \end{array}\right] = \frac{1}{9}\int u^{-4}\,du = \frac{1}{9}\,\frac{u^{-3}}{-3} + C =$$

$$= -\frac{1}{27\sin^3(\theta)} = -\frac{(9+x^2)^{3/2}}{27x^3} + C$$



$\sin(\theta) = \dfrac{x}{\sqrt{9+x^2}}$

För integraler där integranden är en kvot med polynom av $\sin(\theta)$ och $\cos(\theta)$ kan ett variabelbyte $x = \tan\left(\frac{\theta}{2}\right)$ skriva om detta till en rationell funktion med polynom av x i täljare och nämnare. Variabelbytet ger

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1}{\frac{1}{\cos^2\left(\frac{\theta}{2}\right)}} = \frac{1}{1+\tan^2\left(\frac{\theta}{2}\right)} = \frac{1}{1+x^2}$$

$$\cos(\theta) = \cos\left(2\cdot\frac{\theta}{2}\right) = 2\cos^2\left(\frac{\theta}{2}\right) - 1 = \frac{2}{1+x^2} - 1 = \frac{1-x^2}{1+x^2}$$

$$\sin(\theta) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = 2\tan\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right) = \frac{2x}{1+x^2}$$

$$dx = \frac{1}{2}\frac{1}{\cos^2\left(\frac{\theta}{2}\right)}\,d\theta \Rightarrow d\theta = 2\cos^2\left(\frac{\theta}{2}\right)dx = \frac{2}{1+x^2}\,dx$$

Ex 11

$$\int \frac{1}{2+\cos(\theta)}\,d\theta = \begin{bmatrix} x = \tan\left(\frac{\theta}{2}\right) \\ \cos(\theta) = \frac{1-x^2}{1+x^2} \\ d\theta = \frac{2}{1+x^2}\,dx \end{bmatrix} = \int \frac{1}{2+\frac{1-x^2}{1+x^2}} \cdot \frac{2}{1+x^2}\,dx =$$

$$= \int \frac{2}{2+2x^2+1-x^2}\,dx = 2\int \frac{1}{x^2+3}\,dx = \frac{2}{3}\int \frac{1}{\left(\frac{x}{\sqrt{3}}\right)^2+1}\,dx$$

$$= \frac{2}{3}\sqrt{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\left(\frac{\theta}{2}\right)\right) + C$$

Ex:

$$\int \frac{3}{6+2\sin(\theta)}\,d\theta = \begin{bmatrix} x = \tan\left(\frac{\theta}{2}\right) \\ \sin(\theta) = \frac{2x}{1+x^2} \\ d\theta = \frac{2}{1+x^2}\,dx \end{bmatrix} = \int \frac{3}{6+\frac{4x}{1+x^2}} \cdot \frac{2}{1+x^2}\,dx =$$

$$= \int \frac{6}{6+6x^2+4x}\,dx = \int \frac{1}{x^2+\frac{2}{3}x+1}\,dx = \int \frac{1}{x^2+2\cdot\frac{1}{3}x+\frac{1}{9}-\frac{1}{9}+1}\,dx$$

$$= \int \frac{1\cdot\frac{9}{8}}{\left(\left(x+\frac{1}{3}\right)^2+\frac{8}{9}\right)\cdot\frac{9}{8}}\,dx = \frac{9}{8}\int \frac{1\cdot\frac{\sqrt{8}}{3}}{\left(\frac{3}{\sqrt{8}}\left(x+\frac{1}{3}\right)\right)^2+1}\,\frac{3}{\sqrt{8}}\,dx = \begin{bmatrix} u = \frac{3}{\sqrt{8}}\left(x+\frac{1}{3}\right) \\ du = \frac{3}{\sqrt{8}}\,dx \end{bmatrix} =$$

$$= \frac{9}{8}\cdot\frac{\sqrt{8}}{3}\int \frac{1}{u^2+1}\,du = \frac{3}{\sqrt{8}}\tan^{-1}(u) + C = \frac{3}{\sqrt{8}}\tan^{-1}\left(\frac{3}{\sqrt{8}}\left(x+\frac{1}{3}\right)\right) + C =$$

$$= \frac{3}{\sqrt{8}}\tan^{-1}\left(\frac{3}{\sqrt{8}}\left(\tan\left(\frac{\theta}{2}\right)+\frac{1}{3}\right)\right) + C$$

4

För integraler som innehåller $\sqrt[n]{ax+b}$ kan ett variabelbyte $ax+b = u^n$ förenkla.

Ex 8

$$\int \frac{1}{1+\sqrt{2x}}\,dx = \left[\begin{matrix} 2x = u^2 \\ x = \frac{1}{2}u^2 \\ dx = udu \end{matrix}\right] = \int \frac{u}{1+u}\,du = \int \frac{1+u \;\; -1}{1+u}\,du =$$

$$= \int \left(1 - \frac{1}{1+u}\right)du = u - \ln(|1+u|) + C = \underline{\underline{\sqrt{2x} - \ln(1+\sqrt{2x}) + C}}$$

Variabelbyte för att elliminera både $\sqrt[3]{}$ och $\sqrt{}$

Ex 10

$$\int \frac{1}{x^{1/2}(1+x^{1/3})}\,dx = \left[\begin{matrix} x = u^6 \\ dx = 6u^5 \end{matrix}\right] = \int \frac{6u^5}{u^3(1+u^2)}\,dx =$$

$$= 6\int \frac{1+u^2 -1}{1+u^2}\,du = 6\int \left(1 - \frac{1}{1+u^2}\right)du = 6\left(u - \tan^{-1}(u)\right) + C =$$

$$= \underline{\underline{6\,(x^{1/6} - \tan^{-1}(x^{-1/6}) + C}}$$