## Buckinghams Pi-sats

För varje fysikalisk lag

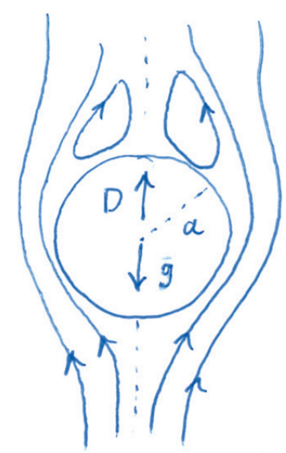
$$f(q_1, q_2, \dots, q_n) = 0$$

som relaterar storheterna $q_1, q_2, \dots, q_n$ så finns en ekvivalent lag

$$g(\pi_1, \pi_2, \dots, \pi_m) = 0 \qquad (m \leq n)$$

där $\pi_1, \pi_2, \dots, \pi_m$ är det minsta antal oberoende dimensionslösa storheter som kan bildas av $q_1, q_2, \dots, q_n$.

**Ex:** En kula faller fritt i en viskös vätska. Efter en viss tid får den konstant hastighet $U$ [m/s]. Hur kan vi bestämma $U$?




- $a$ [m] kulans radie
- $\rho_0$ [kg/m³] kulans densitet
- $\rho$ [kg/m³] vätskans densitet
- $\mu$ [Pa·s = kg·m⁻¹·s⁻¹] vätskans viskositet
- $D$ [N = kg·m·s⁻²] dragkraften
- $g$ [m·s⁻²] tyngdaccelerationen

Summa krafter likamed 0, ty $U$ = konstant, så att

$$\boxed{D = m \cdot g = \frac{4\pi a^3}{3} \cdot \rho_0 \cdot g = \frac{4\pi a^3 \rho_0 g}{3}} \qquad (*)$$

Antag att det finns ett fysikaliskt samband

$$f(a, \rho_0, U, \mu, D) = 0$$

Dimensionsanalys:

$$
\begin{aligned}
a &\sim L \\
\rho &\sim M \cdot L^{-3} \\
U &\sim L\, T^{-1} \\
\mu &\sim M L^{-1} T^{-1} \\
D &\sim M\, L\, T^{-2}
\end{aligned}
$$

Fråga: Hur många dimensionslösa kvantiteter kan vi bilda av $a, \rho, U, \mu, D$?

Dvs hur många oberoende dimensionslösa storheter

$$q = a^{\alpha_1} \rho^{\alpha_2} U^{\alpha_3} \mu^{\alpha_4} D^{\alpha_5}$$

kan vi bilda?

Vi vill välja $\alpha_1, \alpha_2, \dots, \alpha_5$ så att

$$q \sim L^{\alpha_1} \cdot (M L^{-3})^{\alpha_2} (L T^{-1})^{\alpha_3} (M L^{-1} T^{-1})^{\alpha_4} (M L T^{-2})^{\alpha_5}$$

$$= L^{\alpha_1 - 3\alpha_2 + \alpha_3 - \alpha_4 + \alpha_5} T^{-\alpha_3 - \alpha_4 - 2\alpha_5} M^{\alpha_2 + \alpha_4 + \alpha_5} = L^0 T^0 M^0$$

$$\Updownarrow$$

$$
\begin{cases}
\alpha_1 - 3\alpha_2 + \alpha_3 - \alpha_4 + \alpha_5 = 0 \\
\qquad\qquad -\alpha_3 - \alpha_4 - 2\alpha_5 = 0 \\
\qquad \alpha_2 \qquad + \alpha_4 + \alpha_5 = 0
\end{cases}
\quad \sim
$$

$$\sim \begin{array}{c} \begin{matrix} (\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5) \end{matrix} \\ \left(\begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & -2 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{array}\right) \end{array} \sim \left(\begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & -2 & 0 \end{array}\right) \begin{array}{l} (1)+(3) \\ \sim \\ \cdot(-1) \end{array}$$

$$\sim \left(\begin{array}{ccccc|c} 1 & -3 & 0 & -2 & -1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \end{array}\right) \overset{(1)+3\cdot(2)}{\sim} \begin{array}{c} \begin{matrix} (\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5) \end{matrix} \\ \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \end{array}\right) \end{array} \Longleftrightarrow$$

$$\Longleftrightarrow \begin{cases} \alpha_4 = s \\ \alpha_5 = t \\ \alpha_3 = -\alpha_4 - 2\alpha_5 = -s-2t \\ \alpha_2 = -\alpha_4 - \alpha_5 = -s-t \\ \alpha_1 = -\alpha_4 - 2\alpha_5 = -s-2t \end{cases} \Longleftrightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} = s \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

Alltså maximalt två oberoende dimensionslösa variabler, t ex

$$\pi_1 = a^1 \rho^1 U^1 \mu^{-1} = \frac{\rho a U}{\mu} = Re \quad (\text{Reynolds tal}) \qquad (s=-1, t=0)$$

$$\pi_2 = a^{-2} \rho^{-1} U^{-2} D^1 = \frac{D}{\rho a^2 U^2} \qquad (s=0, t=1)$$

$\pi$-satsen ger ett dimensionslöst fysikaliskt samband

$$0 = \Phi(\pi_1, \pi_2) = \Phi\left(Re, \frac{D}{\rho a^2 U^2}\right)$$

Implicita funktionssatsen ger den att

$$\frac{D}{\rho a^2 U^2} = f(Re)$$

$$D = \rho a^2 U^2 f(Re) = \mu a U \boxed{\frac{\rho a U}{\mu} f(Re)} = \mu a U \tilde{f}(Re)$$

$$\frac{\rho a U}{\mu} f(Re) \overset{\text{def}}{=} \tilde{f}(Re)$$

Approximativ analytisk lösning av Navier-Stokes ekvationer ger att

$$\tilde{f}(Re) = 6\pi\left(1 + \frac{3}{8} Re + \frac{9}{40} Re^2 \log(Re) + O(Re^2)\right)$$

För $Re \ll 1$ får vi

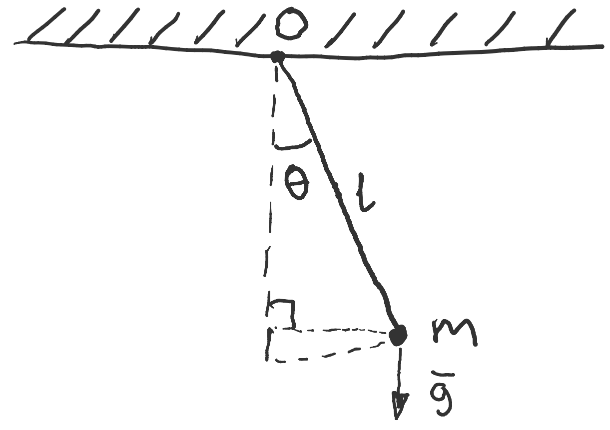
$$D \approx 6\pi \mu a U$$

$$U \approx \frac{D}{6\pi \mu a} \overset{(*)}{=} \frac{\cancel{4\pi}^{\,2} \cancel{a^3}^{\,2} \rho_0 g}{3} \cdot \frac{1}{\cancel{6\pi}_{\,3} \mu \cancel{a}} = \frac{2 a^2 \rho_0 g}{9 \mu}$$

$$\boxed{\underline{\underline{\text{SVAR:}}} \quad U \approx \frac{2 a^2 \rho_0 g}{9 \mu}}$$

Enhetskontroll: $U \sim \dfrac{\cancel{L^2} \cdot \cancel{M} \cdot \cancel{L^{-3}} \cdot L \cdot T^{-2}}{\cancel{M} \cancel{L^{-1}} T^{-1}} = L T^{-1}$

Enkel pendel




- $l$ [m] pendelns längd
- $g$ [m/s²] tyngdacceleration
- $\theta$ [−] vinkel, $\theta = \theta(t)$
- $m$ [kg] massa
- $t$ [s] tid

Total energi

$$E = T + V = \frac{1}{2} m (l\dot{\theta})^2 + mg(l - l\cos(\theta))$$

$$0 = \frac{dE}{dt} = \cancel{\tfrac{1}{2}} m l^2 \cancel{2}\dot{\theta}\ddot{\theta} + mgl\sin(\theta)\dot{\theta} = ml\dot{\theta}(l\ddot{\theta} + g\sin(\theta))$$

Begynnelsevärdesproblem (BVP)

$$\begin{cases} l\ddot{\theta} + g\sin(\theta) = 0 & (1) \\ \theta(0) = \theta_0 & (2) \\ \dot{\theta}(0) = \omega_0 & (3) \end{cases}$$

In parametrar

$l \sim L$

$g \sim L \cdot T^{-2}$

$\theta_0 \sim 1$

$\omega_0 \sim T^{-1}$

Karakteristisk tid:

$$\frac{1}{\omega_0} \sim T \quad \text{eller} \quad \tau = \sqrt{\frac{l}{g}} \sim \left(\frac{\cancel{L}}{\cancel{L} \cdot T^{-2}}\right)^{\frac{1}{2}} = T$$

Dimensionslösa variabler

$$\hat{t} = \frac{t}{\tau} \qquad \hat{\theta} = \hat{\theta}(\hat{t}) = \frac{\theta(t)}{\theta_0},$$

dvs $\boxed{\theta = \theta_0 \hat{\theta}}$

$$\boxed{\text{Låt } \dot{\hat{\theta}} \overset{\text{def}}{=} \frac{d\hat{\theta}}{d\hat{t}}}$$

$$\dot{\theta} = \theta_0 \frac{d\hat{\theta}}{dt} = \theta_0 \frac{d\hat{\theta}}{d\hat{t}} \cdot \frac{d\hat{t}}{dt} = \frac{\theta_0}{\tau} \dot{\hat{\theta}}$$

$$\ddot{\theta} = \frac{\theta_0}{\tau^2} \ddot{\hat{\theta}} = \frac{\theta_0 g}{l} \ddot{\hat{\theta}}$$

$$\begin{aligned} &(1) \Rightarrow \\ &(2) \Rightarrow \\ &(3) \Rightarrow \end{aligned} \begin{cases} \cancel{l}\frac{\theta_0 \cancel{g}}{\cancel{l}} \ddot{\hat{\theta}} + \cancel{g}\sin(\theta_0 \hat{\theta}) = 0 \\ \hat{\theta}(0) = \frac{\theta(0)}{\theta_0} = 1 \\ \frac{\theta_0}{\tau} \dot{\hat{\theta}}(0) = \omega_0 \end{cases} \Rightarrow$$

BVP i dimensionslös form

$$\begin{cases} \theta_0 \ddot{\hat{\theta}} + \sin(\theta_0 \hat{\theta}) = 0 \\ \hat{\theta}(0) = 1 \\ \dot{\hat{\theta}}(0) = \frac{\tau\omega_0}{\theta_0} \end{cases}$$

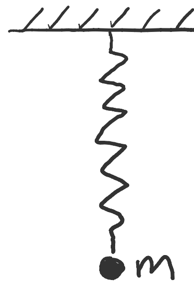
Om vi har $\theta_0 = \varepsilon$ och $\omega_0 = 0$ så får vi

$$\begin{cases} \ddot{\hat{\theta}} + \varepsilon^{-1}\sin(\varepsilon\hat{\theta}) = 0 \\ \hat{\theta}(0) = 1 \\ \dot{\hat{\theta}}(0) = 0 \end{cases}$$

# M7018M F02b Skalning sid 4

den 4 augusti 2020 11:41

## Uppgift 1.2.4



För $x$ = avvikelse från jämviktsläget $x=0$ fås begynnelsevärdesproblemet

$$\begin{cases} m\ddot{x} = -a\,x\,|\dot{x}| - kx \\ x(0) = 0 \\ \dot{x}(0) = V \end{cases} \qquad (1)$$

$m \sim M$ massa [kg]
$v \sim L T^{-1}$ hastighet vid $t=0$ [m/s]
$a \sim M L^{-1} T^{-1}$ dämpningskonstant [$kg \cdot m \cdot s^{-1}$]
$k \sim M \cdot T^{-2}$ fjäderkonstant [$kg \cdot s^{-2}$]

### Karakteristisk tid

$$\tau = m^{\alpha_1} v^{\alpha_2} a^{\alpha_3} k^{\alpha_4} \sim M^{\alpha_1} (LT^{-1})^{\alpha_2} \cdot (ML^{-1}T^{-1})^{\alpha_3} \cdot (MT^{-2})^{\alpha_4} =$$

$$= M^{\alpha_1+\alpha_3+\alpha_4} L^{\alpha_2-\alpha_3} T^{-\alpha_2-\alpha_3-2\alpha_4} = M^0 L^0 T^1$$

$$\Updownarrow$$

$$\begin{cases} \alpha_1 \quad + \alpha_3 + \alpha_4 = 0 \\ \alpha_2 - \alpha_3 = 0 \\ -\alpha_2 - \alpha_3 - 2\alpha_4 = 1 \end{cases} \iff \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 \end{array}\right) \begin{array}{l} \\ \\ (3)+(2) \end{array} \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & -2 & 1 \end{array}\right) \begin{array}{l} (1)+\frac{1}{2}(3) \\ (2)-\frac{1}{2}(3) \\ \cdot(-\frac{1}{2}) \end{array} \sim$$

$$\sim \begin{array}{c} (\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4) \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & 1 & -\frac{1}{2} \end{array}\right) \end{array} \Longrightarrow \begin{cases} \alpha_4 = s \\ \alpha_3 = -\frac{1}{2} - \alpha_4 = -\frac{1}{2} - s \\ \alpha_2 = -\frac{1}{2} - \alpha_4 = -\frac{1}{2} - s \\ \alpha_1 = \frac{1}{2} \end{cases} \Rightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} - s \\ -\frac{1}{2} - s \\ s \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$s=0$ ger $\tau_1 = m^{1/2} v^{-\frac{1}{2}} a^{-1/2} k^0 = \sqrt{\frac{m}{av}} \sim T$

$\Pi_1 = m^0 v^{-1} a^{-1} k^1 = \frac{k}{av} \sim 1$

Låt $\tau_2 = \frac{\tau_1}{\sqrt{\Pi_1}} = \sqrt{\frac{m}{\cancel{av}} \cdot \frac{\cancel{av}}{k}} = \sqrt{\frac{m}{k}}$

### Karakteristisk längd

$$l = m^{\alpha_1} v^{\alpha_2} a^{\alpha_3} k^{\alpha_4} \sim M^{\alpha_1} (LT^{-1})^{\alpha_2} \cdot (ML^{-1}T^{-1})^{\alpha_3} \cdot (MT^{-2})^{\alpha_4} =$$

$$= M^{\alpha_1+\alpha_3+\alpha_4} L^{\alpha_2-\alpha_3} T^{-\alpha_2-\alpha_3-2\alpha_4} = M^0 L^1 T^0$$

$$\Updownarrow$$

$$\begin{cases} \alpha_1 \quad + \alpha_3 + \alpha_4 = 0 \\ \alpha_2 - \alpha_3 = 1 \\ -\alpha_2 - \alpha_3 - 2\alpha_4 = 0 \end{cases} \iff \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & -1 & -1 & -2 & 0 \end{array}\right) \begin{array}{l} \\ \\ (3)+(2) \end{array} \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & -2 & -2 & 1 \end{array}\right) \begin{array}{l} (1)+\frac{1}{2}(3) \\ (2)-\frac{1}{2}(3) \\ \cdot(-\frac{1}{2}) \end{array} \sim$$

$$\sim \begin{array}{c} (\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4) \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 & -\frac{1}{2} \end{array}\right) \end{array} \Longrightarrow \begin{cases} \alpha_4 = s \\ \alpha_3 = -\frac{1}{2} - \alpha_4 = -\frac{1}{2} - s \\ \alpha_2 = \frac{1}{2} - \alpha_4 = \frac{1}{2} - s \\ \alpha_1 = \frac{1}{2} \end{cases} \Rightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} - s \\ -\frac{1}{2} - s \\ s \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$s=0 \Rightarrow l_1 = m^{1/2} v^{\frac{1}{2}} a^{-1/2} k^0 = \sqrt{\frac{mv}{a}} \sim L$

$\Pi_1 = \frac{k}{av}$ som ovan

$l_2 = l_1 / \sqrt{\Pi_1} = \sqrt{\frac{mv}{\cancel{a}} \cdot \frac{\cancel{a}v}{k}} = \sqrt{\frac{mv^2}{2}} \sim L$

Om fjäderkraften << dämpande kraften så väljer vi en tids- och längdskala som inte beror av k, d vs $\tau_1 = \sqrt{\frac{m}{aV}}$ och $l_1 = \sqrt{\frac{mV}{a}}$

## Dimensionslösa variabler

$$\tau = \frac{t}{\tau_1}, \quad \xi(\tau) = \frac{x(t)}{l_1} \quad \text{med} \quad \dot\xi \overset{def}{=} \frac{d\xi}{d\tau}, \quad \underbrace{\tau_1 = \sqrt{\frac{m}{aV}} \text{ och } l_1 = \sqrt{\frac{mV}{a}}}$$

$$\Downarrow$$

$$l_1 \tau_1 = \sqrt{\frac{m}{a\cancel{V}} \cdot \frac{m\cancel{V}}{a}} = \frac{m}{a}$$

$$\dot x = \frac{d}{dt}(l_1 \xi) = l_1 \frac{d\xi}{d\tau} \cdot \frac{d\tau}{dt} = \frac{l_1}{\tau_1}\dot\xi$$

$$\ddot x = \frac{l_1}{\tau_1^2}\ddot\xi$$

Insättning i (1) ger

$$\begin{cases} m\frac{\cancel{l_1}}{\tau_1^2}\ddot\xi = -a l_1 \xi \left|\frac{\cancel{l_1}}{\tau_1}\dot\xi\right| - k\cancel{l_1}\xi \iff \ddot\xi = -\underbrace{\frac{l_1\tau_1^{\cancel{2}}}{m}}_{=1} a\, \xi \left|\frac{1}{\cancel{\tau_1}}\dot\xi\right| - \frac{k\tau_1^2}{m}\xi, \quad \frac{k\tau_1^2}{m} = \frac{k}{\cancel{m}} \cdot \frac{\cancel{m}}{aV} = \Pi_1 \\ \qquad \Updownarrow \\ \ddot\xi = -\xi|\dot\xi| - \Pi_1 \xi \\ \xi(0) = 0 \\ \dot\xi(0) = \frac{\tau_1}{l_1}\dot x(0) = \frac{\tau_1 V}{l_1} V = \sqrt{\frac{\cancel{m}}{\cancel{a}V}\frac{\cancel{a}}{\cancel{m}V}}\, V = \frac{V}{V} = 1 \end{cases}$$

Om vi sätter $\varepsilon = \frac{k}{aV} = \Pi_1$ så får vi det dimensionslösa skalade problemet

$$\begin{cases} \ddot\xi = -\xi|\dot\xi| - \varepsilon\xi \\ \xi(0) = 0 \\ \dot\xi(0) \quad 1 \end{cases}$$