Metod för att visa entydighet hos lösningar till ett randvärdesproblem.

Ex. Visa att problemet

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f(x) & i\ \Omega \times (0,T) \\ u(x,0) = g(x) & i\ \Omega \\ u = h(x) & på\ \partial\Omega \times (0,T) \end{cases}$$

har högst en lösning $u(x,t)$. Anta att det finns två lösningar u_1 och u_2. Sätt $\tilde{u}(x,t) = u_1(x,t) - u_2(x,t)$. Då uppfyller $\tilde{u}$:

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t} - \Delta \tilde{u} = 0 & i\ \Omega \times (0,T) \\ \tilde{u}(x,0) = 0 & i\ \Omega \\ \tilde{u} = 0 & på\ \partial\Omega \times (0,T) \end{cases} \qquad (\square)$$

Multiplicera ekv. med $\tilde{u}$ och integrera över Ω $\Rightarrow$

$$\int_\Omega \left(\frac{\partial \tilde{u}}{\partial t} - \Delta \tilde{u}\right)\tilde{u}\,dx = 0 \Rightarrow \int_\Omega \frac{\partial \tilde{u}}{\partial t}\tilde{u}\,dx - \underbrace{\int_\Omega (\Delta \tilde{u})\tilde{u}\,dx}_{\text{Part int, Lemma 1, F12}} = 0$$

$$\int_\Omega \frac{\partial}{\partial t}\left(\frac{1}{2}\tilde{u}^2\right)dx + \int_\Omega \nabla\tilde{u}\cdot\nabla\tilde{u}\,dx + \underbrace{\int_{\partial\Omega} \tilde{u}(\nabla\tilde{u}\cdot\hat{n})\,dS}_{=0 \text{ pg.a } \tilde{u}=0 \text{ på } \partial\Omega} = 0$$

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\tilde{u}^2\,dx + \int_\Omega |\nabla\tilde{u}|^2\,dx = 0$$

Låt

$$E(t) = \int_\Omega \frac{1}{2}\tilde{u}^2\,dx \qquad (\text{"Energin"})$$

Då gäller $E(t) \geq 0$ och

$$\frac{dE}{dt} = -\int_\Omega |\nabla\tilde{u}|^2\,dx \leq 0$$

$$\Rightarrow E(t) \leq E(0) = \int_\Omega \frac{1}{2}(\tilde{u}(x,0))^2\,dx \overset{(\square)}{=} 0$$

Alltså $E(t) = 0 \quad \forall\, t \in [0,T] \Rightarrow$

$\tilde{u}(x,t) = 0 \Rightarrow u_1 = u_2$ v.s.v.

Ex. Visa att problemet

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0 & i\ \Omega \times (0,T) \\ u(x,0) = 0 & i\ \Omega \\ \dfrac{\partial u}{\partial t}(x,0) = 0 & i\ \Omega \\ u = 0 & på\ \partial\Omega \times (0,T) \end{cases} \qquad (*)$$

endast har lösningen $u = 0$.

1) Uppenbarligen är $u = 0$ en lösning.

2) Antag att det finns en lösning $u \neq 0$. Definiera energin hos lösningen som

$$E(t) = \int_\Omega \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}|\nabla u|^2\,dx.$$

Av begynnelsevillkoren följer att $E(0) = 0$. Dessutom

$$\frac{dE}{dt} = \frac{d}{dt}\int_\Omega \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}|\nabla u|^2\,dx$$

$$= \int_\Omega \underbrace{\frac{\partial}{\partial t}\left(\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2\right)}_{\frac{1}{2}\cdot 2\frac{\partial u}{\partial t}\cdot\frac{\partial^2 u}{\partial t^2}} + \underbrace{\frac{\partial}{\partial t}\left(\frac{1}{2}|\nabla u|^2\right)}_{\frac{1}{2}\cdot 2\nabla u\cdot\frac{\partial}{\partial t}(\nabla u)}\,dx$$

För $v(t) \in \mathbb{R}^n$ är $\frac{d}{dt}|v|^2 = \frac{d}{dt}(v^T v) = (v^T)' v + v^T v' = 2v^T v' = 2\bar{v}\cdot v'$

$$= \int_\Omega \frac{\partial^2 u}{\partial t^2}\cdot\frac{\partial u}{\partial t} + \underbrace{\nabla u\cdot\nabla\left(\frac{\partial u}{\partial t}\right)}_{\Downarrow\ \text{Partiell integration (Lemma 1, F12)}}\,dx$$

$$= \underbrace{\int_\Omega \frac{\partial^2 u}{\partial t^2}\cdot\frac{\partial u}{\partial t} - \Delta u\cdot\frac{\partial u}{\partial t}\,dx}_{\underbrace{\left(\frac{\partial^2 u}{\partial t^2} - \Delta u\right)}_{=0 \text{ enligt } (*)}\frac{\partial u}{\partial t}} + \int_{\partial\Omega} \frac{\partial u}{\partial t}(\nabla u\cdot\hat{n})\,dS$$

$$= \int_{\partial\Omega} \underbrace{\frac{\partial u}{\partial t}}_{=0}(\nabla u\cdot\hat{n})\,dS$$

eftersom $u = 0$ på $\partial\Omega \times (0,T) \Rightarrow \frac{\partial u}{\partial t} = 0$ på $\partial\Omega \times (0,T)$

$= 0$

Alltså $E(t) = E(0) = 0 \Rightarrow \frac{\partial u}{\partial t} = 0,\ \nabla u = 0 \Rightarrow$

$\Rightarrow \boxed{u = 0}$ v.s.v.

Antag att vi vill bestämma extremaler till funktionalen

$$J(u) = \int_\Omega L(x, u, \nabla u)\, dx \quad u \in C^2(\bar{\Omega}).$$

där $\Omega \subset \mathbb{R}^n$ och $u : \bar{\Omega} \to \mathbb{R}$.

$$J(u+h) - J(u) = \int_\Omega \underbrace{L(x, u+h, \nabla u + \nabla h) - L(x, u, \nabla u)}_{\frac{\partial L}{\partial u} h + \sum_{i=1}^n \frac{\partial L}{\partial u_{x_i}} \frac{\partial h}{\partial x_i} + O(|h|^2 + |\nabla h|^2)}\, dx$$

Alltså blir differential av J i "punkten" u:

$$\delta_u J(h) = \int_\Omega \frac{\partial L}{\partial u} h + \sum_{i=1}^n \frac{\partial L}{\partial u_{x_i}} \frac{\partial h}{\partial x_i}\, dx$$

Om $h \in C_c^\infty(\Omega)$ kan vi integrera partiellt

$$\delta_u J(h) = \int_\Omega \left(\frac{\partial L}{\partial u} - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\partial L}{\partial u_{x_i}} \right) \right) h\, dx$$

$$\delta_u J(h) = 0 \quad \forall\, h \in C_c^\infty(\Omega) \Rightarrow$$

$$\boxed{\frac{\partial L}{\partial u} - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\partial L}{\partial u_{x_i}} \right) = 0 \text{ i } \Omega} \quad \text{(Euler-Lagrange)}$$

Ex. Dirichlets princip: Antag att u minimerar funktionalen

$$J(u) = \int_\Omega \frac{1}{2} |\nabla u|^2 - f(x) u\, dx$$

över alla u s.a. $u = 0$ på $\partial\Omega$. Då är u en lösning till

$$\begin{cases} -\Delta u = f(x) & \text{i } \Omega \\ u = 0 & \text{på } \partial\Omega. \end{cases}$$

Bevis. $L(x, u, \nabla u) = \frac{1}{2} |\nabla u|^2 - f(x) u$.

Euler-Lagrange:

$$\frac{\partial L}{\partial u} - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\partial L}{\partial u_{x_i}} \right) = 0 \Rightarrow$$

$$-f(x) - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\partial u}{\partial x_i} \right) = 0 \Rightarrow$$

$$-f(x) - \Delta u = 0 \quad \text{v.s.v.}$$

Ex. Egenvärdesproblemet: Antag att u minimerar funktionalen

$$J(u) = \int_\Omega \frac{1}{2} |\nabla u|^2 dx$$

över alla u s.a. $u = 0$ på $\partial\Omega$ och

$$\int_\Omega |u|^2 dx = 1 \quad \text{(bivillkor)}.$$

Då uppfyller u

$$\begin{cases} -\Delta u = \lambda u & \text{i } \Omega \\ u = 0 & \text{på } \partial\Omega. \end{cases}$$

Bevis. För att hantera bivillkoret sätt

$$L(x, u, \nabla u) = \frac{1}{2} |\nabla u|^2 - \frac{\lambda}{2} u^2$$

Euler-Lagrange:

$$\frac{\partial L}{\partial u} - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\partial L}{\partial u_{x_i}} \right) = 0 \Rightarrow$$

$$-\lambda u - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\partial u}{\partial x_i} \right) = 0 \Rightarrow$$

$$-\lambda u - \Delta u = 0 \quad \text{v.s.v.}$$

Ex. Vågekvationen: Antag att u minimerar funktionalen

$$J(u) = \int_0^T \int_\Omega \frac{1}{2} \rho u_t^2 - \frac{c^2}{2} |\nabla u|^2 dx\, dt$$

över alla u s.a $u = 0$ på $\partial\Omega \times (0, T)$.

$$L(t, x, u, u_t, \nabla u) = \frac{1}{2} \rho u_t^2 - \frac{c^2}{2} |\nabla u|^2$$

$$\frac{\partial L}{\partial u} - \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial u_t} \right) - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{\partial L}{\partial u_{x_i}} \right) = 0 \quad \text{(Euler-Lagrange)}$$

$$0 - \frac{\partial}{\partial t} (\rho u_t) - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(-c^2 \frac{\partial u}{\partial x_i} \right) = 0 \Rightarrow$$

$$-\rho \frac{\partial^2 u}{\partial t^2} + c^2 \Delta u = 0 \quad \text{(vågekvationen!)}$$