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# MNA Circuit Equations for the PEEC Method using Coupled Current Sources

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### Introduction

This report details the frequency- and time- domain modified nodal analysis (MNA) circuit equations using coupled current sources for the capacitive/electric field coupling, Fig. 1. The report is a continuation of: J. Ekman, *Derivation of MNA Circuit Equations for the PEEC Method*, UAq/LTU Research Report, Sep. 2003. This formulation is equal to the MNA-PEEC circuit equation formulation derived by Ruehli and Garrett [1]. Other formulations uses for example coupled voltage sources, as shown in Fig. 2

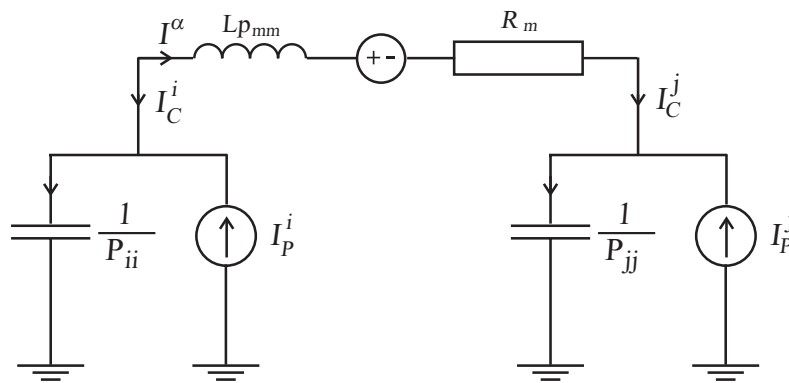


Figure 1: Basic PEEC circuit using coupled current sources for capacitive/electric field coupling.

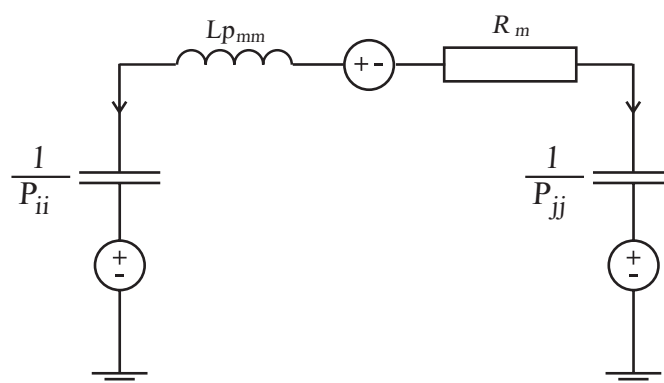


Figure 2: Basic PEEC circuit using coupled voltage sources for capacitive/electric field coupling.



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### The MNA FD PEEC Circuit Equations

#### KVL, Kirchoffs voltage law

The application of KVL to the basic PEEC cell in Fig. 1 results in

$$\mathbf{A}\mathbf{V} + (\mathbf{R} + j\omega\mathbf{L})\mathbf{I} = \mathbf{V}_S \quad (1)$$

where  $\mathbf{R}$  is a diagonal matrix containing the d.c. resistance between the nodes,  $\mathbf{L}$  contains the partial inductances [2, 3], and  $\mathbf{V}_S$  is a voltage source excitation vector.

#### KCL, Kirchoffs current law

The application of KCL to each no of the basic PEEC cell in Fig. 1 results in

$$\mathbf{A}^T\mathbf{I} = \mathbf{I}_C \quad (2)$$

where:

$\mathbf{A}$  is a  $n_L \times n_c$  matrix containing the connectivity information, -1 leaving node, +1 entering node.

$\mathbf{I}$  is  $n_L \times 1$  vector containing the branch currents (solution).

$\mathbf{I}_C$  is  $n_C \times 1$  vector containing the current for the capacitive/electric field coupling.

$\mathbf{I}_C$  can be further expressed as, for each node:

$$\mathbf{I}_C = j\omega\mathbf{C}_{self}\mathbf{V} - \mathbf{I}_P \quad (3)$$

where:

$\mathbf{C}_{self}$  is the *pseudo – capacitance* to each node,  $C_{nn} = \frac{1}{p_{nn}}$ .

$\mathbf{I}_P$  is the current controlled current source (CCCS) modelling the capacitive/electric field coupling from other surface cells.  $\mathbf{I}_P = \sum_{\forall i, i \neq j} \frac{p_{ij}}{ii} \mathbf{I}_C$  where  $p_{ij}$  is complex for frequency domain modeling.

Combining (2) and (3) gives:

$$\mathbf{A}^T\mathbf{I} = j\omega\mathbf{C}_{self}\mathbf{V} - \mathbf{I}_P \quad (4)$$

and by using the CCCS description, (4) turns into

$$\mathbf{A}^T\mathbf{I} = j\omega\mathbf{C}_{self}\mathbf{V} - \sum_{\forall i, i \neq j} \frac{p_{ij}}{ii} \mathbf{I}_C \quad (5)$$

and by using the relationship in (2) and by re-arranging, (5) can be written as

$$j\omega\mathbf{C}_{self}\mathbf{V} + \mathbf{P}_{norm}\mathbf{A}^T\mathbf{I} = 0 \quad (6)$$

where:



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$\mathbf{P}_{norm}$  is a  $n_c \times n_c$  matrix containing the normalized coefficient of potentials, with elements of the type:

- -1, for  $i=j$ ,
- $-\frac{p_{ij}}{p_{ii}}$ , for  $i \neq j$ .

The resulting circuit equations are, by combining (1) and (6)

$$\begin{bmatrix} j\omega\mathbf{C}_{self} & \mathbf{P}_{norm}\mathbf{A}^T \\ \mathbf{A} & \mathbf{R} + j\omega\mathbf{L} \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{I} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{V}_S \end{bmatrix}. \quad (7)$$

To summarize the matrices:

$\mathbf{C}_{self}$  is a  $n_c \times n_c$  diagonal matrix containing the *pseudo*-capacitances,  $C_{ii}$ .

$\mathbf{P}_{norm}$  is a  $n_c \times n_c$  matrix containing the normalized coefficient of potentials, with elements of the type:

- -1, for  $i=j$ ,
- $-\frac{p_{ij}}{p_{ii}}$ , for  $i \neq j$ .

$\mathbf{A}$  is a  $n_L \times n_c$  matrix containing the connectivity information.

$\mathbf{L}$  is a  $n_L \times n_L$  matrix containing the partial inductances.

$\mathbf{V}$  is  $n_C \times 1$  vector containing is the node voltages (solution).

$\mathbf{I}$  is  $n_L \times 1$  vector containing the branch currents (solution).

$\mathbf{V}_S$  is  $n_L \times 1$  vector containing the voltage source excitation.

The addition of an external current source under the previous assumptions effects the charge of the directly excited patch, but also the charges of the surrounding patched (through the mutual electric field coupling effects described by the coefficients of potentials). This require the updating of the equation system (r.h.s.) into

$$\begin{bmatrix} j\omega\mathbf{C}_{self} & \mathbf{P}_{norm}\mathbf{A}^T \\ \mathbf{A} & \mathbf{R} + j\omega\mathbf{L} \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{I} \end{bmatrix} = \begin{bmatrix} -\mathbf{P}_{norm}\mathbf{I}_S \\ \mathbf{V}_S \end{bmatrix}. \quad (8)$$

The equation system in (8) is refereed to as the modified nodal analysis (MNA) equation system as presented in [4, 5, 1, 6].



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### The MNA TD PEEC Circuit Equations

The time domain equations for the solution of PEEC model problems can be derived in the same way as for the frequency domain. The difference is the inclusion of the time retardation that is expressed as a complex part in the partial mutual couplings for frequency domain models, Eq. (9), while for time domain PEEC models the retardation is written as a finite time delay, Eq. (10). The two equations below Eq. (9) and (10), displays the induced voltage in cell  $\alpha$  due to the magnetic field coupling with cell  $\beta$ .

$$V_{\alpha\beta} = j\omega Lp_{\alpha\beta} I_{\beta} e^{-j\omega\tau_{\alpha\beta}} \quad (9)$$

$$v_{\alpha\beta} = Lp_{\alpha\beta} \frac{di_{\beta}(t - \tau_{\alpha\beta})}{dt} \quad (10)$$

### Quasi-Static, TD MNA Equations

For quasi-static time domain PEEC models, the MNA system can be written as

$$\begin{bmatrix} \mathbf{C}_{self} \frac{d}{dt} & \mathbf{P}_{norm} \mathbf{A}^T \\ \mathbf{A} & -(\mathbf{R} + \mathbf{L} \frac{d}{dt}) \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{I} \end{bmatrix} = \begin{bmatrix} -\mathbf{P}_{norm} \mathbf{I}_S \\ \mathbf{V}_S \end{bmatrix} \quad (11)$$

Discretizing matrix Eq. (11) in time by using the Backward Euler (BE) scheme yields

$$\begin{bmatrix} \mathbf{C}_{self} \frac{1}{\Delta t} & \mathbf{P}_{norm} \mathbf{A}^T \\ \mathbf{A} & -(\mathbf{R} + \mathbf{L} \frac{1}{\Delta t}) \end{bmatrix} \begin{bmatrix} \mathbf{V}_n \\ \mathbf{I}_n \end{bmatrix} = \begin{bmatrix} -\mathbf{P}_{norm} \mathbf{I}_S + \mathbf{C}_{self} \mathbf{V}_{n-1} \frac{1}{\Delta t} \\ \mathbf{V}_S + \mathbf{L} \mathbf{I}_{n-1} \frac{1}{\Delta t} \end{bmatrix} \quad (12)$$

where for a fixed time step,  $\Delta t$ , the leftmost block only has to be calculated once.

### Full-wave, TD MNA Equations

The full-wave solution of PEEC models must include the time retarded electric- and magnetic-field couplings. The frequency domain representation through complex mutual partial elements must be converted to individual time delays in the time domain. This results in real valued matrix entries in the  $\mathbf{L}$  and  $\mathbf{S}$ -matrices and the delayed coupling of the corresponding currents and/or potentials. Comparing the electric field coupling for the time- and frequency-domain for a PEEC one-cell shows the difference.

$$\begin{array}{ll} \text{Frequency domain} \rightarrow & Lp_{\alpha\beta} e^{-j\omega\tau_{\alpha\beta}} \quad j\omega \quad I_{\beta}(\omega) \\ \text{Time domain} \rightarrow & Lp_{\alpha\beta} \quad \frac{d}{dt} \quad i_{\beta}(t - \tau_{\alpha\beta}) \end{array}$$

As can be seen, the time retardation expressed as a phase shift incorporated in the (complex) mutual partial elements in the FD is expressed as a real time retardation in the coupled current



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in the TD. This implies that each mutual partial element has its own coupled current(s) due to the different delay times, and the matrix-vector-products, involving the  $\mathbf{L}$ - and  $\mathbf{S}$ -matrices, in the previous equations are converted to matrix(row)-matrix(column)-products in the full-wave time domain formulation.

Since the self terms in the  $\mathbf{L}$ - and  $\mathbf{P}_{norm}$ -matrices are not retarded the derivation of the full-wave equations is simplified if the self terms are separated from the mutual terms in the  $\mathbf{L}$ - and  $\mathbf{P}_{norm}$ -matrices according to

$$\mathbf{L} = \mathbf{L}_S + \mathbf{L}_M \quad (13)$$

$$\mathbf{P}_{norm} = -\mathbf{1} + \mathbf{P}_{norm_M} \quad (14)$$

where  $\mathbf{1}$  is the identity matrix. Substituting Eq. (13) and (14) into (11) yields

$$\begin{bmatrix} \mathbf{C}_{self} \frac{d}{dt} & (-\mathbf{1} + \mathbf{P}_{norm_M}) \mathbf{A}^T \\ \mathbf{A} & -(\mathbf{R} + (\mathbf{L}_S + \mathbf{L}_M) \frac{d}{dt}) \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{I} \end{bmatrix} = \begin{bmatrix} (\mathbf{1} - \mathbf{P}_{norm_M}) \mathbf{I}_S \\ \mathbf{V}_S \end{bmatrix} \quad (15)$$

Since the mutual coupled terms depend on previous node voltages and branch currents they are considered to be known and therefore moved to the RHS. Further, discretizing Eq. (15) using backward/forward Euler, as in Eq. (12), the equation system is easily solved.

The full-wave solution requires the storage of previous nodal voltages  $\mathbf{V}(t)$ , branch currents  $\mathbf{I}_L(t)$ , and current source excitation  $\mathbf{I}_S(t)$  to enable the evaluation of the individually retarded electric and magnetic coupling terms.



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