Research Report

PEEC State Variable Formulation

&

Stability Analysis

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Introduction

This report details the equations governing the PEEC State Variable Formulation for an Admittance and a Modified Nodal Analysis (MNA) Method presented in previous research reports.

Basic stability analysis for the continuous (Ordinary Differential Equations and Neutral Delay Differential Equations) and discretized (Backward Euler) system is also detailed. The electromagnetic quasi-static equations (ODE's) are separated from the full wave equations (NDDE's) for clarity.

EM Quasi-static PEEC State Variable Formulation

By writing the PEEC circuit equations in state variable form facilitates the stability analysis of PEEC models. This section presents the PEEC circuit equations and the corresponding state variable formulation for the EM quasi-static case (when no retarded(!) couplings are considered) for both the Admittance and MNA method. For the full-wave case presented in the next section only the MNA representation is given.

Admittance method

The Admittance method circuit equations the state variable formulation is based on is taken from previous research reports and repeated here for clarity, Eq. (1).

$$\begin{bmatrix} -\mathbf{A} & -(\mathbf{R}+\mathbf{L}\frac{d}{dt}) \\ \mathbf{P}^{-1}\frac{d}{dt}+\mathbf{Y}_L & -\mathbf{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{I}_L \end{bmatrix} = \begin{bmatrix} \mathbf{V}_S \\ \mathbf{I}_S \end{bmatrix} \tag{1}$$

The state variable formulation require Eq. (1) to be recast into the following form

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \mathbf{A}_s\mathbf{x}+\mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x}+\mathbf{D}\mathbf{u} \end{aligned} \tag{2}$$

where $\mathbf{A}_s$ is used in the state variable formulation to avoid mixup with the connectivity matrix $\mathbf{A}$ used in the circuit equations. The state variable formulation is possible by defining

$$\mathbf{x} = \begin{bmatrix} \mathbf{I}_L \\ \mathbf{V} \end{bmatrix} \tag{3}$$

$$\mathbf{A}_s = \begin{bmatrix} -\mathbf{L}^{-1}\mathbf{R} & -\mathbf{L}^{-1}\mathbf{A} \\ \mathbf{P}\mathbf{A}^T & -\mathbf{P}\mathbf{Y}_L \end{bmatrix} \tag{4}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} & -\mathbf{L}^{-1} \\ \mathbf{P} & \mathbf{0} \end{bmatrix} \tag{5}$$

$$\mathbf{u} = \begin{bmatrix} \mathbf{I}_S \\ \mathbf{V}_S \end{bmatrix} \tag{6}$$

Modified nodal analysis method

The MNA method circuit equations the state variable formulation is based on is taken from previous research reports and repeated here for clarity, Eq. (7).

$$\begin{bmatrix} -\mathbf{A} & -(\mathbf{R}+\mathbf{L}\frac{d}{dt}) \\ \mathbf{F}\frac{d}{dt}+\mathbf{S}^T\mathbf{Y}_L & -\mathbf{S}^T\mathbf{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{I}_L \end{bmatrix} = \begin{bmatrix} \mathbf{V}_S \\ \mathbf{S}^T\mathbf{I}_S \end{bmatrix} \tag{7}$$

The state variable formulation in Eq. (2) is possible by defining

$$\mathbf{x} = \begin{bmatrix} \mathbf{I}_L \\ \mathbf{V} \end{bmatrix} \tag{8}$$

$$\mathbf{A}_s = \begin{bmatrix} -\mathbf{L}^{-1}\mathbf{R} & -\mathbf{L}^{-1}\mathbf{A} \\ \mathbf{F}^{-1}\mathbf{S}^T\mathbf{A}^T & -\mathbf{F}^{-1}\mathbf{S}^T\mathbf{Y}_L \end{bmatrix} \tag{9}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} & -\mathbf{L}^{-1} \\ \mathbf{F}^{-1}\mathbf{S}^T & \mathbf{0} \end{bmatrix} \tag{10}$$

$$\mathbf{u} = \begin{bmatrix} \mathbf{I}_S \\ \mathbf{V}_S \end{bmatrix} \tag{11}$$

Full-wave MNA PEEC State Variable Formulation

The state variable formulation for full-wave PEEC system include the retarded electric and magnetic field couplings. For the full-wave formulation the circuit equations require the separation of the self and mutual terms resulting in a MNA circuit equation system according to

$$\begin{bmatrix} -\mathbf{A} & -(\mathbf{R}+(\mathbf{L}_S+\mathbf{L}_M)\frac{d}{dt}) \\ \mathbf{F}\frac{d}{dt}+(\mathbf{S}_S+\mathbf{S}_M)\mathbf{Y}_L & -(\mathbf{S}_S+\mathbf{S}_M)\mathbf{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{I}_L \end{bmatrix} = \begin{bmatrix} \mathbf{V}_S \\ (\mathbf{S}_S+\mathbf{S}_M)\mathbf{I}_S \end{bmatrix} \tag{12}$$

The state variable formulation for PEEC full-wave systems, results in neutral delay differential equations, is written in the following form

$$\begin{aligned} \mathbf{y}\prime(t) &= \mathbf{A}_S\mathbf{y}(t)+\mathbf{E}\mathbf{y}(t-\tau)+\mathbf{F}\mathbf{y}\prime(t,t-\tau), & t \geq t_0 \\ \mathbf{y}(t) &= \mathbf{B}\mathbf{u}(t), & t \leq t_0 \end{aligned} \tag{13}$$

taken from [1].

The state variable form is possible by defining

$$\mathbf{y} = \begin{bmatrix} \mathbf{I}_L \\ \mathbf{V} \end{bmatrix} \tag{14}$$

$$\mathbf{y\prime} = \begin{bmatrix} \mathbf{I}_L(t-\tau) \\ \mathbf{V}(t-\tau) \end{bmatrix} \tag{15}$$

$$\mathbf{A}_s = \begin{bmatrix} -\mathbf{L}_S{}^{-1}\mathbf{R} & -\mathbf{L}_S{}^{-1}\mathbf{A} \\ \\ \mathbf{F}^{-1}\mathbf{S}_S{}^T\mathbf{A}^T & -\mathbf{F}^{-1}\mathbf{S}_S{}^T\mathbf{Y}_L \end{bmatrix} \tag{16}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} & -\mathbf{L}_S{}^{-1} \\ \\ \mathbf{F}^{-1}(\mathbf{S}_S + \mathbf{S}_M) & \mathbf{0} \end{bmatrix} \tag{17}$$

$$\mathbf{E} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \\ \mathbf{F}^{-1}\mathbf{S}_M\mathbf{A}^T & \mathbf{F}^{-1}\mathbf{S}_M\mathbf{Y}_L \end{bmatrix} \tag{18}$$

$$\mathbf{F} = \begin{bmatrix} -\mathbf{L}_S{}^{-1}\mathbf{L}_M & \mathbf{0} \\ \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{19}$$

$$\mathbf{u} = \begin{bmatrix} \mathbf{I}_S(t, t-\tau) \\ \mathbf{V}_S(t, t-\tau) \end{bmatrix} \tag{20}$$

Stability Analysis

The stability problem for PEEC models, and other integral equation formulated EM analysis methods, is well known and often termed *late time instability*. The instabilities are mainly seen in the time domain as a undamped oscillation starting at some 'late' time totally masking the real solution, see Fig. 1. This section summarizes theory for ODE's and DDE's, and observations on stability for quasi-static and full-wave PEEC time domain simulations.

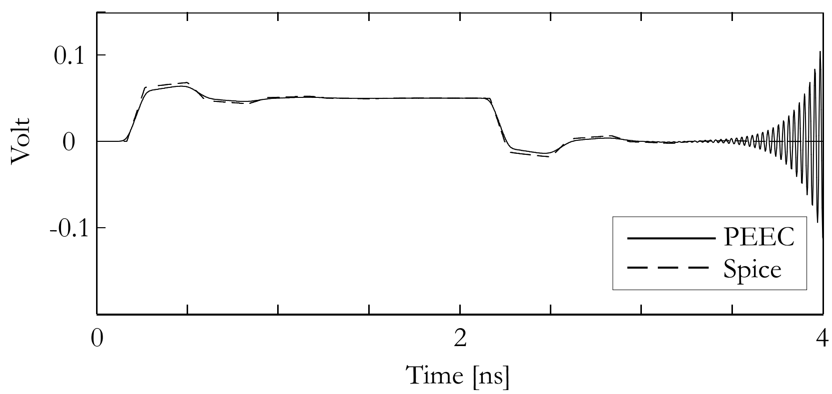

Figure 1: Late time instability for a PEEC model.

The stability analysis for the PEEC method can be carried out for the continuous or discretized system and is therefore presented in two separate sections. It is important to note that the stability of the numerical (discretized system) solution is related to, but not the same as stability of the continuous system (ODE or DDE).

Continuous system

For the quasi-static PEEC method the time domain circuit equations are generic first order ordinary differential equations (ODE's). The stability of this type of equations, and the corresponding numerical solution, is well known and documented. For the full-wave case the time domain equations are neutral delay differential equations (NDDE's) and the stability of these are more demanding than for ODE's.

Ordinary Differential Equations

The system to be studied is in the following form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{21}$$

where, depending on the formulation, the **A**-matrix is given by Eq. (4) or (9). For the ODE in Eq. (21) to be stable the solution has to converge as $t \to \infty$.

The convergence of an ODE can be ensured by finding the eigenvalues λ for the A-matrix and control that they all are located in the left hand plane.

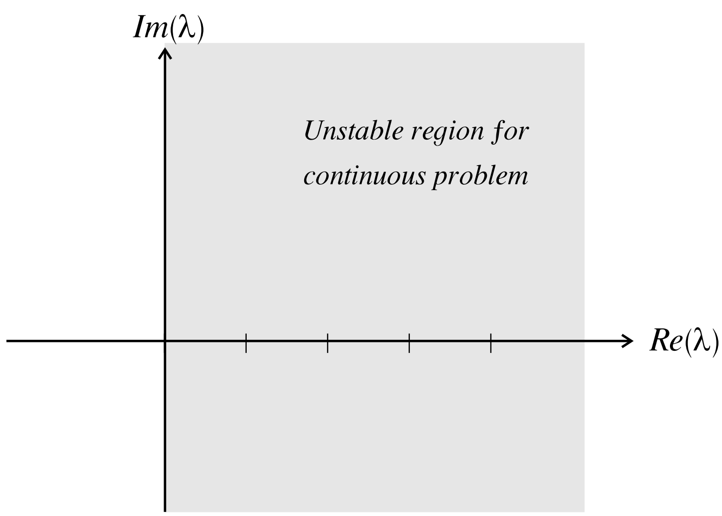

Figure 2: Stability region for continuous problem.

Neutral Delay Differential Equation

The type of DDE's that arise in full-wave PEEC model analysis are entitled neutral differential equations (NDDE's) [2] since the state variable $\dot{\mathbf{y}}$ is dependent on delayed derivative terms, see Eq. (13).

Discretized system

For ODE's the concept of *A-stability* means that the method is asymptotically stable for any time step size $h > 0$ for models with eigenvalues in the left hand plane [3], see previous section. Since, in this report, the discretized systems are obtained using the Backward Euler (BE) method for the time derivatives, the following is true:

The Backward Euler method is an implicit time integration method that is unconditionally stable (for ODE's) as long as the ODE is stable.

Ordinary Differential Equations

Since the BE is used for the time integration in this report, the stability analysis of the discretized ODE's are redundant, see above. However, the discretized system can be studied and stability can be ensured by calculating the eigenvalues for the corresponding system matrices, given below, and control that they are outside the unit circle located at +1.

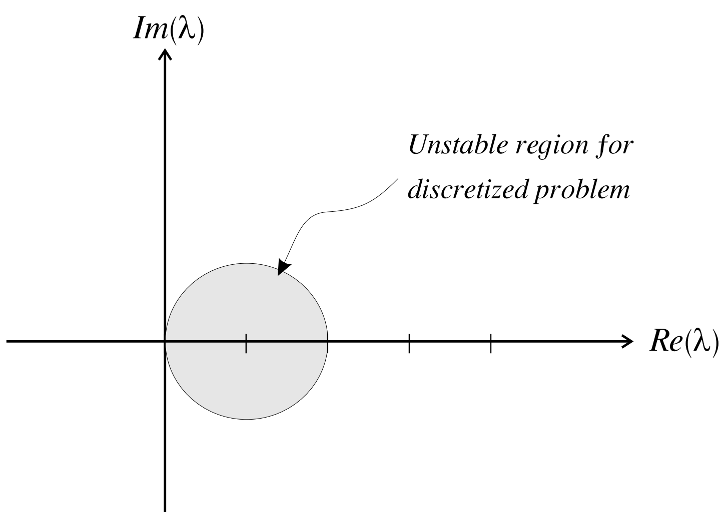

Figure 3: Stability region for discretized problem.

On the next side the discretized quasi-static TD equations are given for the Admittance and MNA method.

Admittance method. The discretized quasi-static PEEC time domain Admittance method equations using BE are

$$\begin{bmatrix} \mathbf{V}_n \\ \\ \mathbf{I}_{nL} \end{bmatrix} = \begin{bmatrix} -\mathbf{A} & -(\mathbf{R} + \mathbf{L}\frac{1}{dt}) \\ \\ \mathbf{P}^{-1}\frac{1}{dt} + \mathbf{Y}_L & -\mathbf{A}^T \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{V}_S - \mathbf{L}\frac{1}{dt}\mathbf{I}_{n-1_L} \\ \\ \mathbf{I}_S + \mathbf{P}^{-1}\frac{1}{dt}\mathbf{V}_{n-1} \end{bmatrix} \tag{22}$$

MNA method. The discretized quasi-static PEEC time domain MNA equations using BE are

$$\begin{bmatrix} \mathbf{V}_n \\ \\ \mathbf{I}_{nL} \end{bmatrix} = \begin{bmatrix} -\mathbf{A} & -(\mathbf{R} + \mathbf{L}\frac{1}{dt}) \\ \\ \mathbf{F}\frac{1}{dt} + \mathbf{S}^T\mathbf{Y}_L & -\mathbf{S}^T\mathbf{A}^T \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{V}_S - \mathbf{L}\frac{1}{dt}\mathbf{I}_{n-1_L} \\ \\ \mathbf{S}^T\mathbf{I}_S + \mathbf{F}\frac{1}{dt}\mathbf{V}_{n-1} \end{bmatrix} \tag{23}$$

Research Report

Delay Differential Equations

Numerical Experiments

3-cell Transmission line

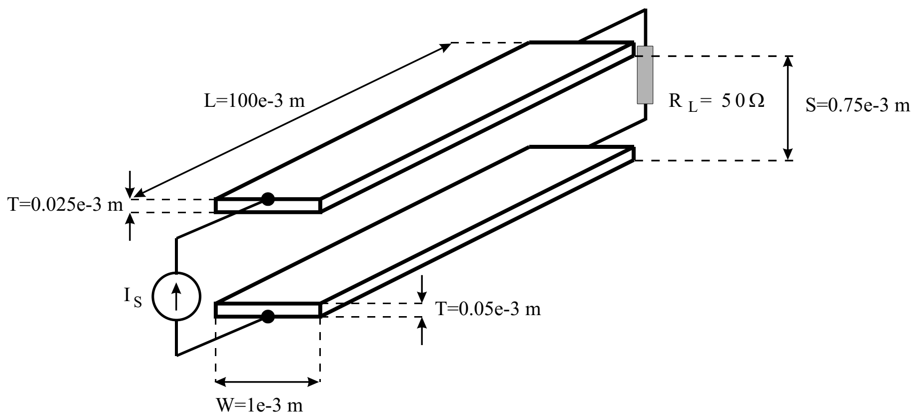

Figure 4: Transmission line used for numerical experiments.

Bibliography

[1] A. Bellen, N. Guglielmi, and A. Ruehli, "Methods for linear systems of circuit delay differential equations of neutral type," *IEEE Trans. Circuits Syst.*, vol. 46, pp. 212–216, Jan. 1999.

[2] C. T. H. Baker, C. A. H. Paul, and D. R. Willé, "Issues on the numerical solution of evolutionary delay differential equations," University of Manchester, Tech. Rep., 1994, numerical Analysis Report No. 28.

[3] A. E. Ruehli, U. Miekkala, A. Bellen, and H. Heeb, "Stable time domain solutions for EMC problems using PEEC circuit models," in *Proc. of the IEEE Int. Symposium on EMC*, Chicage, IL, USA, 1994, pp. 371–376.