$\boxed{151218 - \text{Uppg } 1}$

(a) $\ln e^{1/2} + \ln 2 - \ln \frac{2}{e}$

$= \frac{1}{2} + \ln 2 - (\ln 2 - \underbrace{\ln e}_{=1})$

$= \frac{1}{2} + \ln 2 - \ln 2 + 1$

$= \frac{3}{2}$

(b) $\tan\left(\frac{2\pi}{3}\right) = \frac{\sin\left(\frac{2\pi}{3}\right)}{\cos\left(\frac{2\pi}{2}\right)} = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}$

(c) $\frac{d}{dx}(\cot(x))^2 = 2 \cdot \cot x \cdot \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$

$= 2\cot x \cdot \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x}$

$= -2\frac{\cot x}{\sin^2 x}(\sin^2 x + \cos^2 x) = -2\frac{\cot x}{\sin^2 x}$

(d) $\left(x^2 - \frac{2}{x}\right)^9 = \sum_{k=0}^{9}\binom{9}{k}(x^2)^{9-k}\left(-\frac{2}{x}\right)^k$

$= \sum_{k=0}^{9}\binom{9}{k} x^{2(9-k)} \cdot (-2)^k \cdot x^{-k} = \sum_{k=0}^{9}\binom{9}{k} x^{18-2k-k}(-2)^k$

$= \sum_{k=0}^{9}\binom{9}{k}(-2)^k \cdot x^{18-3k}$

Konstant term då $18-3k=0$

dvs $k=6$ ger konstant

$\binom{9}{6}(-2)^6 = \frac{\overset{3}{\cancel{9}}\cdot\overset{4}{\cancel{8}}\cdot 7\cdot\cancel{6}\cdot\cancel{5}\cdot\cancel{4}}{\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1}2^6 = 84\cdot 64 = 5376$

$\boxed{\text{15 12 18 – Uppg 2}}$

(a) $\displaystyle\lim_{x\to 2} \frac{\sqrt{x}-\sqrt{2}}{x-2} = \left[\frac{0}{0}\right] = \lim_{x\to 2} \frac{(\sqrt{x}-\sqrt{2})(\sqrt{x}+\sqrt{2})}{(x-2)(\sqrt{x}+\sqrt{2})}$

$\displaystyle = \lim_{x\to 2} \frac{\cancel{(x-2)}\cdot 1}{\cancel{(x-2)}(\sqrt{x}+\sqrt{2})} = \lim_{x\to 2} \frac{1}{\sqrt{x}+\sqrt{2}} = \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}}$

(b) $\displaystyle\lim_{x\to -2-} \frac{|x+2|}{x^2-4} = \left[\frac{0}{0}\right] = \lim_{x\to -2-} \frac{-\cancel{(x+2)}}{(x-2)\cancel{(x+2)}}$

$\displaystyle = \lim_{x\to -2-} \frac{-1}{x-2} = \frac{-1}{-2-2} = \frac{1}{4}$

(c) $\displaystyle\lim_{x\to 1} \frac{x^2-3x+2}{x-1} = \left[\frac{0}{0}\right] = \lim_{x\to 1} \frac{\cancel{(x-1)}(x-2)}{\cancel{(x-1)}\cdot 1}$

$\displaystyle = \lim_{x\to 1} x-2 = -1$

$\boxed{151218 - \text{Uppg } 3}$

(a) $f'(x) = x - 1 - \frac{1}{x-1} = \frac{(x-1)^2 - 1}{x-1}$

$$= \frac{x^2 - 2x \cancel{+1} \cancel{-1}}{x-1} = \frac{x(x-2)}{x-1}$$

| | | 0 | | 1 | | 2 | |
|---|---|---|---|---|---|---|---|
| $x$ | $-$ | 0 | $+$ | | $+$ | | $+$ |
| $x-2$ | $-$ | | $-$ | | $-$ | 0 | $+$ |
| $x-1$ | $-$ | | $-$ | 0 | $+$ | | $+$ |
| $f'(x)$ | $-$ | 0 | $+$ | ej. def | $-$ | 0 | $+$ |
| $f(x)$ | $\searrow$ | | $\nearrow$ | | $\searrow$ | | $\nearrow$ |

Växande: $[0, 1) \cup [2, \infty)$

Avtagande: $(-\infty, 0] \cup (1, 2]$

---

(b) $y' = k \cdot y, \quad y = y(t)$

Allmän lösning $y = C \cdot e^{k \cdot t} = C \cdot e^{-\ln(2) \cdot t}$

Villkoret $y(0) = y_0$ ger

$$y_0 = C \cdot \underbrace{e^{-\ln(2) \cdot 0}}_{=1} = C$$

så

$$y = y_0 \cdot e^{-\ln(2) \cdot t}$$

Tid till $y = \frac{y_0}{3}$ ?

$$\frac{y_0}{3} = y(t) = y_0 \cdot e^{-\ln(2) \cdot t}$$

$$\frac{\cancel{y_0}}{3} = \cancel{y_0} \cdot e^{-\ln(2) \cdot t}$$

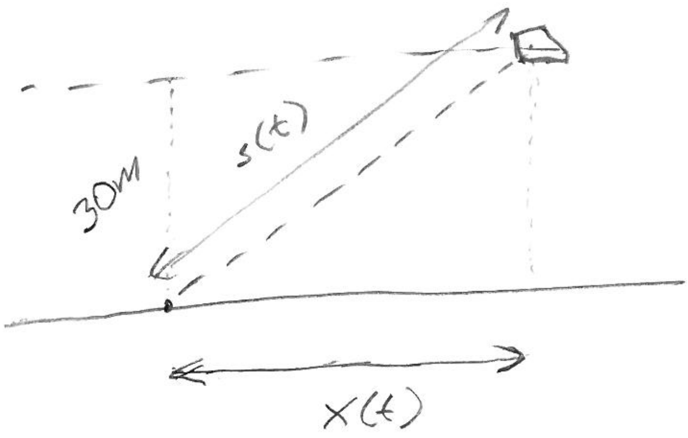
$$\frac{1}{3} = e^{-\ln(2) \cdot t}$$

Tag ln

$$\underbrace{\ln 3^{-1}}_{-\ln(3)} = -\ln(2) \cdot t$$

$$\boxed{t = \frac{\ln 3}{\ln 2}} \leftarrow \text{Svar}$$

151218 – Uppg 4




Tid: $t$ [min]

Markavstånd: $x(t)$ [m]

Längd lina: $s(t)$ [m]

Vid tidpunkten $t = t_0$

är $x(t_0) = 40$ och $x'(t_0) = 10$

Sökes: $s'(t_0)$

Samband:

$$s^2(t) = 30^2 + x^2(t)$$

Derivera map $t$

$$2\,s(t)\cdot s'(t) = 2\,x(t)\cdot x'(t)$$

$$s'(t) = \frac{x(t)}{s(t)}\,x'(t)$$

så

$$s'(t_0) = \frac{x(t_0)}{s(t_0)}\cdot x'(t_0)$$

Vi behöver också $s(t_0)$

$$s^2(t_0) = 30^2 + \underbrace{x^2(t_0)}_{40^2}$$

$$s(t_0)^2 = 2500$$

$$s(t_0) = 50$$

Så

$$s(t_0) = \frac{40}{50}\cdot 10 = 8 \text{ m/min}$$

151218 – Uppg 5

$$f(x) = (x-1)^{2/3} + (x+1)^{2/3}$$

Kontinuerlig på $[-3, 3]$

$$f'(x) = \frac{2}{3}(x-1)^{-\frac{1}{3}} + \frac{2}{3}(x+1)^{-\frac{1}{3}}$$

1. Stationära punkter $f'(x) = 0$

$$\frac{2}{3}(x-1)^{-1/3} + \frac{2}{3}(x+1)^{-1/3} = 0$$

$$(x+1)^{-1/3} = -(x-1)^{-1/3}$$

$$\Downarrow$$

$$x+1 = -x+1$$

$$2x = 0$$

$$\boxed{x = 0}$$

2. Singulära punkter $f'(x)$ ej def.

$x - 1 = 0$ $\quad$ $\boxed{x = 1}$

$x + 1 = 0$ $\quad$ $\boxed{x = -1}$

3. Ändpunkter $\boxed{x = -3}$ $\quad$ $\boxed{x = 3}$

Kontinuerlig funktion på slutet begränsat intervall
Största / minsta värde i punkterna ovan

| $x$ | $f(x)$ | |
|---|---|---|
| $-3$ | $16^{1/3} + 4^{1/3}$ | ← max |
| $-1$ | $4^{1/3}$ | ← min |
| $0$ | $2 = 8^{1/3}$ | |
| $1$ | $4^{1/3}$ | ← min |
| $3$ | $4^{1/3} + 16^{1/3}$ | ← max |

$\boxed{\text{151218 - Uppg 6C}}$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1} \qquad (*)$$

i) Visa (*) då $n=1$

$$V.L. = \frac{1}{1\cdot 2} = \frac{1}{2} \qquad H.L = \frac{1}{1+1} = \frac{1}{2} \qquad \text{ok!}$$

ii) Antag att (*) gäller endast då $n=p$ dvs

$$\sum_{k=1}^{p} \frac{1}{k(k+1)} = \frac{p}{p+1}$$

Visa då att (*) gäller då $n = p+1$

$$V.L = \sum_{k=1}^{p+1} \frac{1}{k(k+1)} = \sum_{k=1}^{p} \frac{1}{k(k+1)} + \frac{1}{(p+1)(p+2)}$$

$$= \frac{p}{p+1} + \frac{1}{(p+1)(p+2)} = \frac{p(p+2)}{(p+1)(p+2)} + \frac{1}{(p+1)(p+2)}$$

$$= \frac{p^2+2p+1}{(p+1)(p+2)} = \frac{(p+1)^{2}}{(p+1)(p+2)} = \frac{p+1}{(p+1)+1} = H.L$$

ok!

iii) Enligt induktionsaxiomet följer av i) & ii) ovan att (*) gäller för alla heltal $n \geq 1$.