Tentamen i Matematik 2: M0030M.

Datum: 2018-06-01
Skrivtid: 09:00–14:00
Antal uppgifter: 6 (**32** poäng)
Examinator: Norbert Euler
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Tillåtna hjälpmedel: Inga

Till alla uppgifterna skall fullständiga lösningar lämnas.
Resonemang och uträkningar ska vara tydligt presenterade.
Även endast delvis lösta problem kan ge poäng.
Enbart svar ger 0 poäng.

Betygsgränser: $14p - 19p = 3$; $20p - 25p = 4$; $26p - 32p$=5.

NOTE: The English version follows after Problem 6 of the Swedish version.

Problem 1:

a) Bestäm medelvärdet av funktionen

$$f(x) = x^3 + 1$$

på intervallet $I = [-1,\ 2]$ och ange de värden i I där funktionen $f(x)$ antar detta medelvärde. [3 poäng]

b) Betrakta funktionen

$$F(x) = x^2 \int_{x^3}^{1} e^{t^2}\, dt$$

och beräkna $\frac{dF}{dx}$. [2 poäng]

Problem 2: Beräkna följande integraler:

a) $\int \cos^4 x \sin^3 x\, dx$ [3 poäng]

b) $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\, dx$ [3 poäng]

Problem 3:

a) Beräkna arean av området som begränsas av kurvorna

$$x = 4y - y^2 \quad \text{och} \quad x = y$$

i den första kvadranten av xy-planet, dvs $x > 0$ och $y > 0$. [2 poäng]

b) Beräkna volymen av rotationskroppen som bildas när området som beskrevs i Problem 3a) roterar kring x-axeln. [3 poäng]

Problem 4: Betrakta följande inhomogena ekvationsystem i variablerna x_1, x_2, x_3, x_4, x_5:

$$\begin{aligned} x_3 + 2x_5 &= 1 \\ x_1 + 6x_2 + 2x_3 + 4x_5 &= -2 \\ x_4 + 5x_5 &= 2. \end{aligned} \tag{1}$$

a) Bestäm alla lösningar till systemet (1). Hur många lösningar har systemet (1)?

[3 poäng]

b) Betrakta nu det motsvarande homogena systemet till (1) och bestäm en linjärt oberoende mängd S av vektorer så att span$\{S\}$ motsvarar alla lösningar till det homogena systemet. Visa uttryckligen att din mängd S är linjärt oberoende.

[2 poäng]

Problem 5: Betrakta avbildningen $T: \mathbb{R}^3 \to \mathbb{R}^2$, så att varje vektor $\mathbf{x} = (x_1, x_2, x_3)$ i $\mathbb{R}^3$ avbildas i $\mathbb{R}^2$ på följandet sätt:

$$T: (x_1, x_2, x_3) \mapsto (x_1 - 5x_2 + 4x_3, x_2 - 6x_3).$$

a) Ange definitionen av en linjär avbildning $S: \mathbb{R}^n \to \mathbb{R}^m$ och visa att den ovan givna avbildningen T är linjär. [2 poäng]

b) Bestäm standardmatrisen till T. [1 poäng]

c) Bestäm värdemängden av T och avgör om T är 1-1 (injektiv) på sin värdemängd. [3 poäng]

Problem 6: Lös ett och endast ett av följande problem a) eller b).

a) Finn en reduktionsformel för följande integral:

$$I_n = \int \sin^n x \, dx, \qquad n = 1, 2, 3, \ldots$$

b) Betrakta matrisekvationen

$$A\mathbf{x} = \mathbf{b},$$

där A är en inverterbar $n \times n$ matris, $\mathbf{x} \in \mathbb{R}^n$ och $\mathbf{b} \in \mathbb{R}^n$. Formulera och bevisa Cramers regel för ekvationen.

English Version:

Problem 1:

a) Find the average of the function

$$f(x) = x^3 + 1$$

on the closed interval $I = [-1,\ 2]$ and give the value(s) in the interval I, where the function achieves this average. [3 points]

b) Consider the function

$$F(x) = x^2 \int_{x^3}^{1} e^{t^2}\, dt$$

and calculate $\frac{dF}{dx}$. [2 points]

Problem 2:

Calculate the following integrals:

a) $\int \cos^4 x \sin^3 x \, dx$ [3 points]

b) $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\, dx$ [3 points]

Problem 3:

a) Calculate the area of the region that is enclosed by the curves

$$x = 4y - y^2 \quad \text{and} \quad x = y$$

in the first quadrant of the xy-plane, i.e. for $x > 0$ and $y > 0$. [2 points]

b) Rotate the plane region described in 3a) about the x-axis and find the volume of the resulting solid of revolution. [3 points]

Problem 4: Consider the following non-homogeneous system of linear algebraic equations in the variables $x_1,\ x_2,\ x_3,\ x_4,\ x_5$:

$$\begin{aligned} &x_3 + 2x_5 = 1 \\ &x_1 + 6x_2 + 2x_3 + 4x_5 = -2 \\ &x_4 + 5x_5 = 2. \end{aligned} \tag{2}$$

a) Find all solutions of system (2). How many solutions does system (2) have? [3 points]

b) Consider now the corresponding homogeneous system to system (2) and find a linearly independent set S of vectors in $\mathbb{R}^5$ such that span$\{S\}$ consists of all the solutions of this homogeneous system of equations. You must show explicitly that your set S of vectors is linearly independent. [2 points]

Problem 5: Consider the transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$, such that every vector $\mathrm{x} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ is mapped to $\mathbb{R}^2$ as follows:

$$T: (x_1, x_2, x_3) \mapsto (x_1 - 5x_2 + 4x_3, x_2 - 6x_3).$$

a) Give the definition of a linear transformation $S: \mathbb{R}^n \to \mathbb{R}^m$ and show that the above given transformation is linear. [2 points]

b) Find the standard matrix for the above transformation T. [1 point]

c) Find the range of T and establish whether T is a one-to-one (injective) transformation on its range.

[3 points]

Problem 6: Solve only one of the following three problems, i.e. a) or b).

a) Find a reduction formula for the integral

$$I_n = \int \sin^n x \, dx, \qquad n = 1, 2, 3, \ldots$$

b) Consider the matrix equation

$$A\mathrm{x} = \mathrm{b},$$

where A is an invertible $n \times n$ matrix, $\mathrm{x} \in \mathbb{R}^n$ and $\mathrm{b} \in \mathbb{R}^n$. Formulate and prove Cramer's Rule for this equation.

[5 points]

MOO30M: (2018-06-01)

PROBLEM 1:

a) $$f_{AVE} = \frac{1}{3}\int_{-1}^{2}(x^3+1)\,dx$$

$$= \frac{1}{3}\left[\frac{1}{4}x^4 + x\right]_{-1}^{2}$$

$$= \frac{1}{3}\left[4+2-\left(\frac{1}{4}-1\right)\right]$$

$$f_{AVF} = \frac{9}{4}$$

Now

$$f(c) = \frac{9}{4}$$

or $$c^3+1 = \frac{9}{4}$$

or $$c = \left(\frac{5}{4}\right)^{1/3}$$

②

b) $$F(x) = x^2 \int_{x^3}^{1} e^{t^2} dt$$

$$= -x^2 \int_{1}^{x^3} e^{t^2} dt$$

$$\frac{dF}{dx} = -x^2 \left[e^{x^6} 3x^2 \right] - 2x \int_{1}^{x^3} e^{t^2} dt.$$

$$\frac{dF}{dx} = 2x \int_{x^3}^{1} e^{t^2} dt - 3x^4 e^{x^6}$$

PROBLEM 2:

a) $\int \cos^4 x \, \sin^2 x \, \sin x \, dx.$

LET $u = \cos x$. THEN

$\frac{du}{dx} = -\sin x$ AND

$$\int \cos^4 x \, (1 - \cos^2 x) \, \sin x \, dx.$$

$$= -\int u^4 (1 - u^2) \, du$$

$$= -\int (u^4 - u^6) \, du.$$

$$= -\left[\tfrac{1}{5} u^5 - \tfrac{1}{7} u^7 \right] + C.$$

$$= \tfrac{1}{7} \cos^7 x - \tfrac{1}{5} \cos^5 x + C.$$

b) NOTE:

$$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

WHERE

$$A = \frac{1}{2}, \quad B = \frac{1}{5}, \quad C = -\frac{1}{10}$$

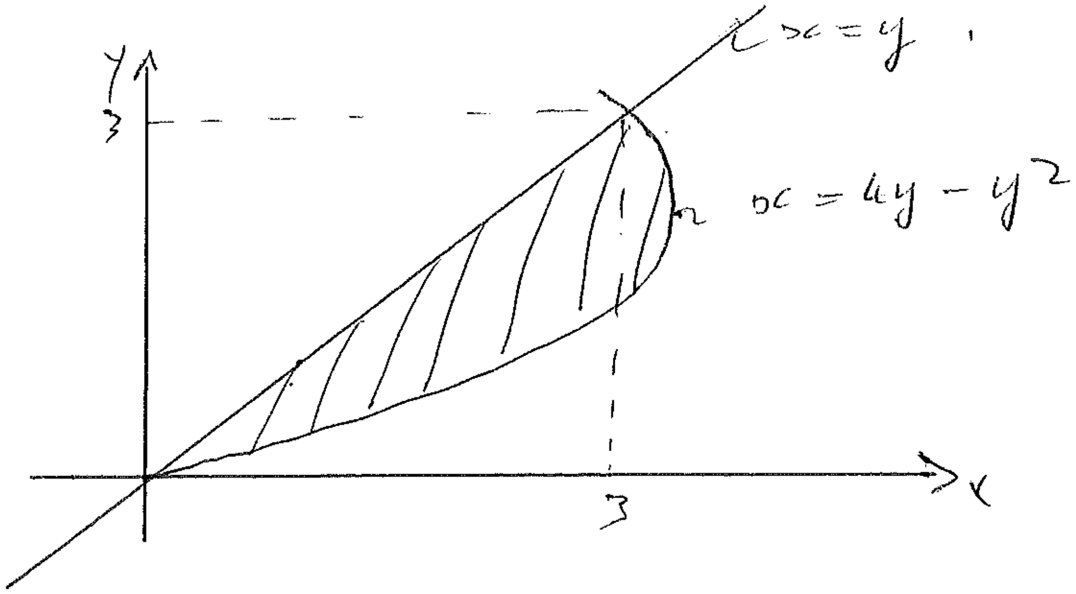
THUS

$$\int \frac{x^2+2x-1}{x(2x-1)(x+2)}\,dx = \frac{1}{2}\int \frac{1}{x}\,dx + \frac{1}{5}\int \frac{1}{2x-1}\,dx$$

$$-\frac{1}{10}\int \frac{1}{x+2}\,dx$$

$$= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C$$

PROBLEM 3: a)



INTERSECTION: $y = 4y - y^2$

so $y=0$, $y=3$.

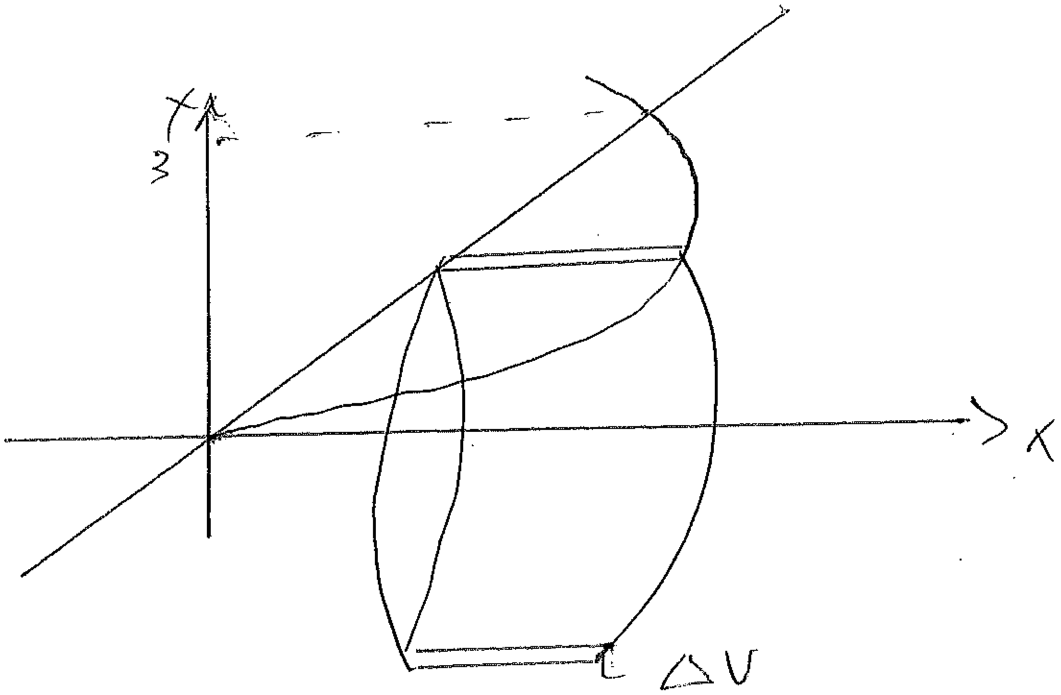
$$\text{AREA} = \int_0^3 (4y - y^2 - y)\,dy$$

$$= \int_0^3 (3y - y^2)\,dy$$

$$= \left[\tfrac{3}{2}y^2 - \tfrac{1}{3}y^3\right]\Big|_0^3$$

$$= \frac{81-54}{6} = \frac{9}{2} \quad \text{SQ. UNITS.}$$

PROBLEM 3 b)



$$\Delta V = 2\pi y\,(4y - y^2 - y)\,\Delta y$$

THUS

$$V = 2\pi \int_0^3 y\,(3y - y^2)\,dy$$

$$= 2\pi \left[y^3 - \frac{1}{4} y^4 \right] \Big|_0^3$$

$$= \frac{27}{2}\pi$$ CUBIC UNITS.

PROBLEM 4:

$$\begin{cases} x_3 + 2x_5 = 1 \\ x_1 + 6x_2 + 2x_3 + 4x_5 = -2 \\ x_4 + 5x_5 = 2. \end{cases}$$

$$\begin{pmatrix} 1 & 6 & 2 & 0 & 4 & -2 \\ 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 5 & 2 \end{pmatrix} \begin{array}{l} \leftarrow \\ \leftarrow (-2)+ \\ \\ \end{array}$$

$$\sim \cancel{\begin{pmatrix} 1 & 6 & 0 & 0 & -4 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & & \end{pmatrix}}$$

$$\sim \begin{pmatrix} \textcircled{1} & 6 & 0 & 0 & 0 & -4 \\ 0 & 0 & \textcircled{1} & 0 & 2 & 1 \\ 0 & 0 & 0 & \textcircled{1} & 5 & 2 \end{pmatrix}$$

THUS $x_2 = t \in \mathbb{R}$; $x_5 = s \in \mathbb{R}$.

AND

$$x_1 = -6t - 4$$

$$x_2 = t$$

$$x_3 = -2s + 1$$

$$x_4 = -5s + 2$$

$$x_5 = s.$$

OR

$$\vec{x} = t\begin{pmatrix}-6\\1\\0\\0\\0\end{pmatrix} + s\begin{pmatrix}0\\0\\-2\\-5\\1\end{pmatrix} + \begin{pmatrix}-4\\0\\1\\2\\0\end{pmatrix} \quad \forall\, s,t \in \mathbb{R}.$$

THUS ∞ - MANY SOLUTIONS.

b) ALL SOLUTIONS FOR THE HOMOGENEOUS SYSTEM ARE

$$\vec{x} = t\begin{pmatrix}-6\\1\\0\\0\end{pmatrix} + s\begin{pmatrix}0\\0\\-2\\-5\\1\end{pmatrix}.$$

THUS

ALL SOLUTIONS = SPAN $\{\vec{u}, \vec{v}\}$,

WHERE

$$\vec{u} = \begin{pmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 0 \\ 0 \\ -2 \\ -5 \\ 1 \end{pmatrix}.$$

$\{\vec{u}, \vec{v}\}$ ARE LINEARLY INDEPENDENT

SINCE

$$c_1 \vec{u} + c_2 \vec{v} = \vec{0}$$

OR

$$\begin{pmatrix} -6c_1 \\ c_2 \\ -2c_2 \\ -5c_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

HAS THE ONLY SOLUTIONS

$c_1 = 0$, $c_2 = 0$.

PROBLEM 5:

a) A TRANSFORMATION $S: \mathbb{R}^n \to \mathbb{R}^m$

IS LINEAR IF

1) $S(\vec{u}+\vec{v}) = S(\vec{u}) + S(\vec{v})$

2) $S(c\vec{u}) = c\, S(\vec{u})$

$\forall\ \vec{u}, \vec{v} \in \mathbb{R}^n \quad \forall\ c \in \mathbb{R}$.

THE GIVEN T IS LINEAR, SINCE

$T: \vec{u} \mapsto T(\vec{u}) = A\vec{u} \quad \forall\ \vec{u} \in \mathbb{R}^3$

WHERE

$$A = \begin{pmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{pmatrix}$$

SO THAT

1) $T(\vec{u}+\vec{v}) = A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$
$= T(\vec{u}) + T(\vec{v})$

a) $T(c\vec{u}) = A(c\vec{u}) = c\,A\vec{u}$

$= c\,T(\vec{u})$.

$\forall\, \vec{u} \in \mathbb{R}^3 \quad \forall\, c \in \mathbb{R}$.

b) $A = \begin{pmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{pmatrix}$.

c). WE CONSIDER $A\vec{x} = \vec{b}$, WHERE

$\vec{x} = (x_1, x_2, x_3)$, $\vec{b} = (b_1, b_2)$.

THEN

$$[A \,|\, \vec{b}] = \begin{pmatrix} 1 & -5 & 4 & b_1 \\ 0 & 1 & -6 & b_2 \end{pmatrix}$$

$$\sim \begin{pmatrix} \textcircled{1} & 0 & -26 & b_1+5b_2 \\ 0 & \textcircled{1} & -6 & b_2 \end{pmatrix}$$

THE SOLUTIONS OF $A\vec{x} = \vec{b}$

ARE THUS

$$\begin{cases} x_1 = 26t + b_1 + 5b_2 \\ x_2 = 6t + b_2 \\ x_3 = t \qquad \forall\, t \in \mathbb{R}. \end{cases}$$

THUS THE TRANSFORMATION IS

ONTO $\mathbb{R}^2$ BUT NOT 1-1.

6q) FIND A REDUCTION FORMULA FOR

$$\int \sin^n x \, dx \quad ; \quad n = 1, 2, 3, \ldots$$

LET $I_n = \int \sin^n x \, dx$.

NOW

$$\int \underbrace{\sin^{n-1} x}_{\uparrow \atop f(x)} \; \underbrace{\sin x}_{\uparrow \atop g'(x)} \, dx$$

: LET $f(x) = \sin^{n-1} x$; $g'(x) = \sin x$

$f'(x) = (n-1)\sin^{n-2} x \cdot \cos x$; $g(x) = -\cos x$

$$\underbrace{\int \sin^{n-1} x \, \sin x \, dx}_{I_n} = -\sin^{n-1} x \cos x + \int (n-1) \sin^{n-2} x \, \underline{\cos^2 x} \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, (\underline{1 - \sin^2 x}) \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \Big[\underbrace{\int \sin^{n-2} x \, dx}_{I_{n-2}} - \underbrace{\int \sin^n x \, dx}_{I_n} \Big]$$

$$\underline{I_n} = -\sin^{n-1} x \cos x + (n-1) I_{n-2} \; \underline{- (n-1) I_n}$$

OR

$$I_n + (n-1) I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$

OR $$\boxed{I_n = -\frac{1}{n} \sin^{n-1} x \cos x + \left(\frac{n-1}{n}\right) I_{n-2}}$$

FOR $n = 2, 3, \ldots$

AND

$$I_1 = \int \sin^1 x \, dx = -\cos x.$$

$$I_0 = \int 1 \, dx = x.$$

THIS IS ALSO IMPORTANT FOR THE EXAM!

6 b) CRAMER'S RULE

LET $A: n \times n$ AND $\vec{b} \in \mathbb{R}^n$. CONSIDER

$$A\vec{x} = \vec{b} \, , \quad \vec{x} \in \mathbb{R}^n .$$

NOTATION:

LET $A_i(\vec{b})$ DENOTE THE MATRIX OBTAINED BY REPLACING THE i^{TH} COLUMN OF A BY THE VECTOR $\vec{b}$.

LET $A = [\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_n]$. THEN

$$A_i(\vec{b}) = [\, \vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_{i-1} \ \underset{\substack{\uparrow \\ i^{TH} \text{ COLUMN}}}{\vec{b}} \ \vec{a}_{i+1} \cdots \vec{a}_n \,] .$$

THEOREM: CRAMER'S RULE

LET $A: n\times n$ BE INVERTIBLE. FOR ANY $\vec{b} \in \mathbb{R}^n$, THE UNIQUE SOLUTION $\vec{x} = (x_1, x_2, \dots, x_n)$ OF

$$A\vec{x} = \vec{b}$$

IS GIVEN BY

$$\boxed{x_i = \frac{\text{DET } A_i(\vec{b})}{\text{DET } A}}, \quad i = 1, 2, \dots, n.$$

PROOF,

CONSIDER $A\vec{x} = \vec{b}$ WITH $A = [\vec{a}_1\ \vec{a}_2 \dots \vec{a}_n] : n\times n$ AND $\vec{b} \in \mathbb{R}^n$. LET'S DENOTE THE $n\times n$ IDENTITY MATRIX BY I. THEN

$$I = [\,\vec{e}_1\ \ \vec{e}_2\ \cdots\ \vec{e}_n\,],$$ WHERE

$\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ IS THE STANDARD BASIS VECTORS OF $\mathbb{R}^n$.

LET $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. BY OUR NOTATION WE HAVE

$$I_i(\vec{x}) = [\,\vec{e}_1\ \ \vec{e}_2\ \cdots\ \overset{i^{TH}\text{ COLUMN}}{\overset{\downarrow}{\vec{x}}}\ \cdots\ \vec{e}_n\,].$$ NOW

$$A\, I_i(\vec{x}) = [\,\underset{=\vec{a}_1}{A\vec{e}_1}\ \ \underset{=\vec{a}_2}{A\vec{e}_2}\ \cdots\ \underset{=\vec{b}}{A\vec{x}}\ \cdots\ \underset{=\vec{a}_n}{A\vec{e}_n}\,].$$

BUT

$$A\vec{e}_1 = [\,\vec{a}_1\ \ \vec{a}_2 \cdots \vec{a}_n\,]\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{a}_1$$

SO

$$A\vec{e}_j = \vec{a}_j. \quad \forall\ j = 1, 2, \dots, n.$$

THUS

$$A\, I_i(\vec{x}) = [\,\vec{a}_1\ \vec{a}_2\ \cdots\ \overset{i^{\text{TH}}\ \text{COLUMN}}{\overset{\downarrow}{\vec{b}}}\ \cdots\ \vec{a}_n\,]$$

SO $$\boxed{A\, I_i(\vec{x}) = A_i(\vec{b})}$$

NOW

$$\text{DET}\left(A I_i(\vec{x})\right) = \text{DET}\left(A_i(\vec{b})\right)$$

OR $$\left(\text{DET } A\right)\left(\text{DET } I_i(\vec{x})\right) = \text{DET}\left(A_i(\vec{b})\right)$$

BUT $\underline{\text{DET } I_i(\vec{x}) = x_i}$: NOTE

$$I_i(\vec{x}) = \begin{bmatrix} 1 & 0 & \cdots & 0 & x_1 & 0 & \cdots & 0 \\ 0 & 1 & & & x_2 & & & 0 \\ \vdots & \vdots & & & x_i & & & \vdots \\ 0 & 0 & \cdots & 0 & x_n & 0 & \cdots & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \cdots & x_1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & x_2 & \vdots & \cdots & 0 \\ \vdots & \vdots & & & & \cdots & 0 \\ \vdots & \vdots & & x_i & 0 & & \\ \vdots & \vdots & & 0 & 1 & \cdots & 0 \\ 0 & 0 & & 0 & 0 & \cdots & 1 \end{bmatrix}$$

NOW

$$\text{DET } I_i(\vec{x}) = (1)(1)\cdots(1)\,x_i\,(1)\cdots(1) = x_i \quad \forall\, i = 1, 2, \ldots, n$$

THUS

$$\text{DET}\,A \; x_i = \text{DET}\,A_i(\vec{b}) \quad \forall \; i = 1, 2, \dots, n.$$

SINCE A IS INVERTIBLE, $\text{DET}\,A \neq 0$, SO THAT

$$x_i = \frac{\text{DET}\,A_i(\vec{b})}{\text{DET}\,A} \quad \forall \; i = 1, 2, \dots, n. \qquad \square$$