$\boxed{150821}$

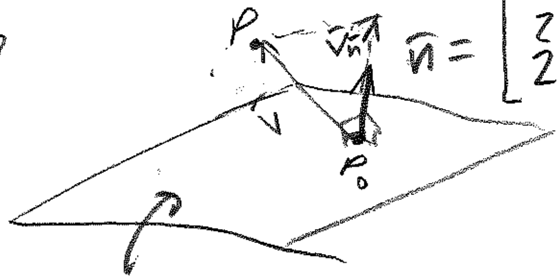
$\boxed{1a}$ Normal $\bar{n} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ passerar $(x, y, z) = (1, 3, 1)$ ger

$$1(x-1) + 1(y-3) + (-1)(z-1) = 0$$

$$x - 1 + y - 3 - z + 1 = 0$$

$$\boxed{x + y - z = 3}$$

$\boxed{1b}$



$\bar{n} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$x + 2y + 2z = 7$

$P = (1, 2, -2)$

P_0 en punkt i planet

t.ex

$P_0 = (7, 0, 0)$

$$\bar{v} = \overrightarrow{P_0 P} = \begin{bmatrix} 1-7 \\ 2-0 \\ -2-0 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ -2 \end{bmatrix}$$

Avståndet

$$d = \|\bar{v}_{\bar{n}}\| = \underset{\substack{\uparrow \\ \text{skalärprod}}}{|s|} = \left| \frac{\bar{v} \bullet \bar{n}}{\|\bar{n}\|} \right| = \left| \frac{\begin{bmatrix} -6 \\ 2 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}{\sqrt{1^2 + 2^2 + 2^2}} \right|$$

$$= \left| \frac{(-6) \cdot 1 + 2 \cdot 2 + (-2) \cdot 2}{\sqrt{9}} \right| = \left| \frac{-6}{3} \right| = 2$$

$\boxed{150821}$

$\boxed{1c}$ Riktning för l_1: $\bar{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

Riktning för l_2: $\bar{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

Riktning vinkelrät mot l_1 och l_2

$$\bar{v} = \bar{v}_1 \times \bar{v}_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} (-1)(-1)-(-2)\cdot 1 \\ -(1(-1)-(-2)\cdot 2) \\ 1\cdot 1-(-1)\cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}$$

Linje med den riktningen genom $(2, -1, 5)$ ges av

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} + t \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}$$

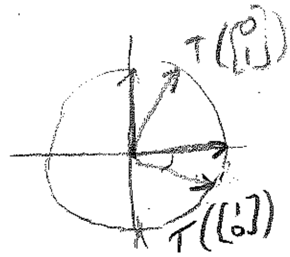
dvs

$$\begin{cases} x = 2 + t\cdot 3 \\ y = -1 - t\cdot 3 \\ z = 5 + t\cdot 3 \end{cases}$$

$\boxed{150821}$ $\boxed{2ai}$

$T_1(\bar{x}) = A_1\bar{x}$ $\quad T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}\sqrt{3}/2\\-1/2\end{bmatrix}$

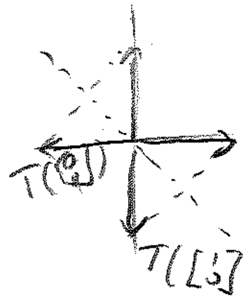
$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1/2\\\sqrt{3}/2\end{bmatrix}$



ger $\quad A_1 = \begin{bmatrix}\sqrt{3}/2 & 1/2\\-1/2 & \sqrt{3}/2\end{bmatrix}$

$T_2(\bar{x}) = A_2\bar{x}$ $\quad T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\-1\end{bmatrix}$

$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\0\end{bmatrix}$



ger $\quad A_2 = \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$

$T_3(\bar{x}) = A_3\bar{x}$ $\quad T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}$

$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$

ger $A_3 = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

Då är

$$T(\bar{x}) = (T_3 \circ T_2 \circ T_1)(x) = \underbrace{A_3 \cdot A_2 \cdot A_1}_{\text{sökt avbildningsmatris}} \cdot \bar{x}$$

$$A = A_3 \cdot A_2 \cdot A_1 = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}\begin{bmatrix}\sqrt{3}/2 & 1/2\\-1/2 & \sqrt{3}/2\end{bmatrix}$$

$$= \begin{bmatrix}0 & 0\\-1 & 0\end{bmatrix}\begin{bmatrix}\sqrt{3}/2 & 1/2\\-1/2 & \sqrt{3}/2\end{bmatrix} = \begin{bmatrix}0 & 0\\-\sqrt{3}/2 & -1/2\end{bmatrix}$$

$\boxed{2aii}$ $\det(A) = 0 \cdot (-1/2) - 0(-\sqrt{3}/2) = 0$ $\quad$ dvs ej inverterbar

150821 | 2b

Bestäm alla $\bar{x}$ så $A\bar{x} = \bar{0}$, dvs lös

$$\begin{bmatrix} 1 & -2 & 1 & 2 & | & 0 \\ 2 & -6 & -3 & 1 & | & 0 \\ -3 & 8 & 2 & -4 & | & 0 \end{bmatrix} \begin{matrix} \\ \leftarrow (-2) \\ \leftarrow (3) \end{matrix} \sim \begin{bmatrix} 1 & -2 & 1 & 2 & | & 0 \\ 0 & -2 & -5 & -3 & | & 0 \\ 0 & 2 & 5 & 2 & | & 0 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow (1) \end{matrix}$$

$$\sim \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & & \end{matrix} \\ \begin{bmatrix} \boxed{1} & -2 & 1 & 2 & | & 0 \\ 0 & \boxed{-2} & -5 & -3 & | & 0 \\ 0 & 0 & 0 & \boxed{-1} & | & 0 \end{bmatrix} \end{array}$$

Bundna: x_1, x_2, x_4

Fri: x_3

$$\boxed{x_3 = t}$$

rad 3: $-x_4 = 0$

$$\boxed{x_4 = 0}$$

rad 2: $-2x_2 - 5\overset{=t}{x_3} - 3\overset{=0}{x_4} = 0$

$$\boxed{x_2 = -\frac{5}{2}t}$$

rad 1: $x_1 - 2\overset{=-\frac{5}{2}t}{x_2} + \overset{=t}{x_3} + 2\overset{=0}{x_4} = 0$

$$x_1 + 5t + t + 0 = 0$$

$$\boxed{x_1 = -6t}$$

dvs

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -6t \\ -5/2\, t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} = \frac{t}{2} \begin{bmatrix} -12 \\ -5 \\ 2 \\ 0 \end{bmatrix} \quad \text{där } t \in \mathbb{R}$$

$\boxed{150821}$ $\boxed{3a}$

$$\int_1^e \underbrace{(x^2+x+1)}_{\uparrow} \ln x \, dx = \Big[\big(\tfrac{1}{3}x^3+\tfrac{1}{2}x^2+x\big)\ln x\Big]_1^e - \int_1^e \big(\tfrac{1}{3}x^3+\tfrac{1}{2}x^2+x\big)\tfrac{1}{x}\,dx$$

$$= \big(\tfrac{1}{3}e^3+\tfrac{1}{2}e^2+e\big)\cdot \overbrace{\ln e}^{=1} - (\ldots)\cdot\overbrace{\ln 1}^{=0} - \int_1^e \big(\tfrac{1}{3}x^2+\tfrac{1}{2}x+1\big)\,dx$$

$$= \tfrac{1}{3}e^3+\tfrac{1}{2}e^2+e - \Big[\tfrac{1}{3}\cdot\tfrac{1}{3}x^3+\tfrac{1}{2}\cdot\tfrac{1}{2}x^2+x\Big]_1^e$$

$$= \tfrac{1}{3}e^3+\tfrac{1}{2}e^2+\cancel{e}-\tfrac{1}{9}e^3-\tfrac{1}{4}e^2-\cancel{e}+\tfrac{1}{9}+\tfrac{1}{4}+1$$

$$= \tfrac{2}{9}e^3+\tfrac{1}{4}e^2+\tfrac{49}{36}$$

$\boxed{3b}$

$$\int \sin^6 x\,dx = I_6 = -\tfrac{1}{6}\sin^5 x\cdot\cos x + \tfrac{5}{6} I_4$$

$$= -\tfrac{1}{6}\sin^5 x\cdot\cos x + \tfrac{5}{6}\Big(-\tfrac{1}{4}\sin^3 x\cos x + \tfrac{3}{4} I_2\Big)$$

Tankerrata:

$$I_2 = \int \sin^2 x\,dx = \int \tfrac{1}{2}(1-\cos 2x)\,dx = \tfrac{1}{2}\big(x - \tfrac{1}{2}\overbrace{\sin 2x}^{2\cdot\sin x\cdot\cos x}\big) + C$$

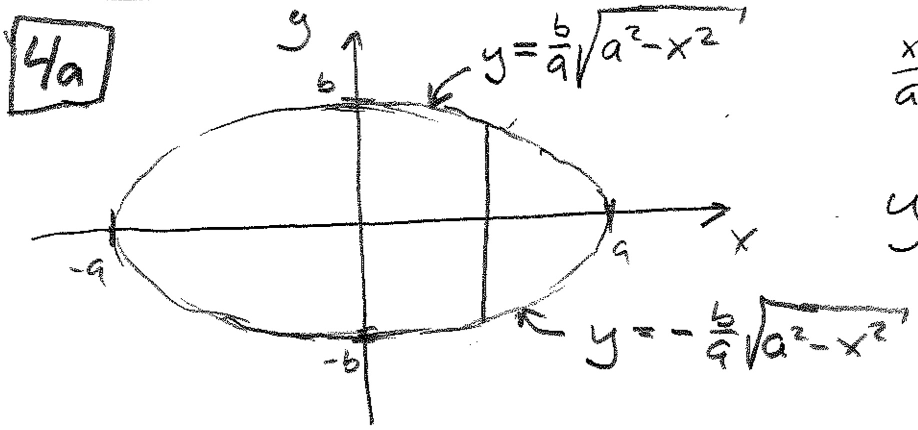
$$= -\tfrac{1}{6}\sin^5 x\cos x - \tfrac{5}{24}\sin^3 x\cos x + \underbrace{\tfrac{15}{24}\cdot\tfrac{1}{2}}_{=5/8}\big(x - \sin x\cos x\big) + D$$

$$= -\tfrac{1}{6}\sin^5 x\cos x - \tfrac{5}{24}\sin^3 x\cos x - \tfrac{5}{16}\sin x\cos x + \tfrac{5}{16}x + D$$

150821 $\boxed{3c}$

$$\frac{d}{dx}\left(\int_1^{\sin x} e^{-t^2}\,dt\right) = e^{-(\sin x)^2}\cdot\frac{d}{dx}\sin x = e^{-\sin^2 x}\cdot\cos x$$

$\boxed{4a}$



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

$$A=\int_{-a}^{a}2\frac{b}{a}\sqrt{a^2-x^2}\,dx = \left[\begin{array}{l|l} x=a\cdot\sin\theta & x=-a \\ \frac{dx}{d\theta}=a\cdot\cos\theta & \theta=-\frac{\pi}{2} \\ & x=a \\ dx=a\cdot\cos\theta\,d\theta & \theta=\frac{\pi}{2}\end{array}\right]$$

$$=2\frac{b}{a}\int_{-\pi/2}^{\pi/2}\sqrt{\underbrace{a^2-a^2\sin^2\theta}_{a^2(1-\sin^2\theta)}}\cdot a\cdot\cos\theta\,d\theta$$

$$=2\cdot\frac{b}{a}\int_{-\pi/2}^{\pi/2}a^2\cdot\cos^2\theta\,d\theta = 2\cdot b\cdot a\int_{-\pi/2}^{\pi/2}\frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$=a\cdot b\left[\theta+\frac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = a\cdot b\left(\frac{\pi}{2}+\overbrace{\frac{1}{2}\sin 2\frac{\pi}{2}}^{=0}-\left(-\frac{\pi}{2}\right)-\overbrace{\frac{1}{2}\sin 2\left(-\frac{\pi}{2}\right)}^{=0}\right)$$

$$=a\cdot b\cdot\pi$$

150821 | 4b

Ellipsens area, höjd z över xy-planet

$$A(z) = z \cdot \sqrt{1-z^2} \cdot \pi$$

ger volym

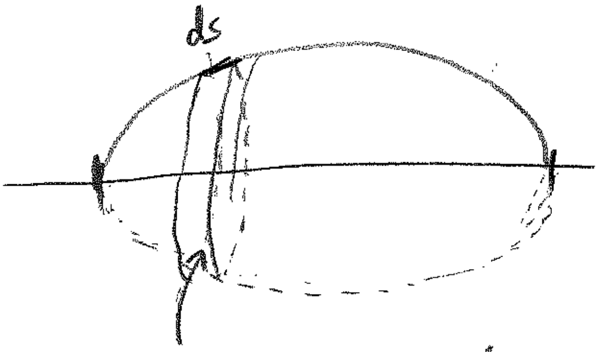
$$V = \int_0^1 A(z)\,dz = \int_0^1 z\sqrt{1-z^2}\,dz = \left[\begin{array}{l|l} u = 1-z^2 & z=0 \\ & u=1 \\ \frac{du}{dz} = -2z & \\ & z=1 \\ -\frac{1}{2}du = z\,dz & u=0 \end{array}\right]$$

$$= \int_1^0 \sqrt{u}\left(-\frac{1}{2}\right)du = \frac{1}{2}\int_0^1 u^{1/2}\,du = \frac{1}{2}\left[\frac{1}{3/2}u^{3/2}\right]_0^1$$

$$= \frac{1}{2}\left(\frac{2}{3}1^{3/2} - \frac{2}{3}0^{3/2}\right) = \frac{1}{3} \text{ v.e.}$$

$\boxed{150821}$ $\boxed{5a}$

$$\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1-\cos\theta) \end{cases}$$



$$dA = 2\pi \cdot y \cdot ds = 2\pi \cdot y \cdot \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \cdot d\theta$$

$$= 2\pi \cdot r(1-\cos\theta)\sqrt{\left(r(1-\cos\theta)\right)^2 + (r\cdot\sin\theta)^2}\, d\theta$$

$$= 2\pi r(1-\cos\theta)\sqrt{r^2(1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{=1})}\, d\theta$$

$$= 2\pi r(1-\cos\theta)\cdot r \cdot \sqrt{2}\sqrt{1-\cos\theta}\, d\theta$$

$$= 2\sqrt{2}\cdot\pi\cdot r^2 (1-\cos\theta)^{3/2} d\theta$$

Area

$$A = \int_{\theta=0}^{\theta=2\pi} dA = \int_0^{2\pi} 2\sqrt{2}\cdot\pi\cdot r^2 (\underbrace{1-\cos\theta}_{2\cdot\sin^2\frac{\theta}{2}})^{3/2} d\theta$$

$$= \int_0^{2\pi} 2\sqrt{2}\cdot\pi\cdot r^2 \cdot 2^{3/2} \left|\sin^3\frac{\theta}{2}\right| d\theta = 2^3 \cdot \pi \cdot r^2 \int_0^{2\pi} \sin^3\frac{\theta}{2}\, d\theta$$

$$= 8\cdot\pi\cdot r^2 \int_0^{2\pi} \sin^2\frac{\theta}{2} \sin\frac{\theta}{2}\, d\theta = 8\pi r^2 \int_0^{2\pi}\left(1-\cos^2\frac{\theta}{2}\right)\sin\frac{\theta}{2}\, d\theta$$

$$\left[\begin{array}{l|l} u = \cos\frac{\theta}{2} & \theta = 0 \\ \frac{du}{d\theta} = -\left(\sin\frac{\theta}{2}\right)\frac{1}{2} & u = 1 \\ & \theta = 2\pi \\ -2\,du = \sin\frac{\theta}{2}\, d\theta & u = -1 \end{array}\right] = 8\pi r^2 \int_1^{-1} (1-u^2)(-2)\, du$$

$$= 16\pi r^2 \int_{-1}^{1} (1-u^2)\, du = 16\pi r^2 \left[u - \frac{1}{3}u^3\right]_{-1}^{1} = 16\pi r^2 \left(\overbrace{1 - \frac{1}{3} - (-1) - \frac{1}{3}(-1)^3}^{2-\frac{2}{3}}\right)$$

$$= \frac{64}{3}\pi r^2$$

150821

5b i

Antag att ekvationssystemet $A\bar{x}=\bar{b}$ har exakt två lösningar $\bar{x}_1 \neq \bar{x}_2$ dvs

$$A\bar{x}_1=\bar{b} \quad \& \quad A\bar{x}_2=\bar{b}$$

Då måste det finnas ytterligare en lösning

$$\bar{x}_3 = \frac{1}{2}\bar{x}_1 + \frac{1}{2}\bar{x}_2$$

ty

$$A\bar{x}_3 = A\left(\frac{1}{2}\bar{x}_1+\frac{1}{2}\bar{x}_2\right) = \frac{1}{2}A\bar{x}_1+\frac{1}{2}A\bar{x}_2$$

$$= \frac{1}{2}\bar{b}+\frac{1}{2}\bar{b} = \bar{b}$$

Det kan alltså inte finnas endast två lösningar.

(Egentligen oändligt med lösningar eftersom

$$\bar{x} = t\cdot\bar{x}_1 + (1-t)\bar{x}_2$$

löser $A\bar{x}=\bar{b}$ för alla $t \in \mathbb{R}$)

5b ii

En linjär avbildning är en avbildning från $\mathbb{R}^n$ till $\mathbb{R}^m$ som uppfyller att

1. $T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v})$
2. $T(\alpha\cdot\bar{u}) = \alpha T(\bar{u})$

För alla $\bar{u},\bar{v} \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$