Tentamen 130326

1a

$$\lim_{x\to 0} \frac{\cos x \cdot \tan x}{x} = \lim_{x\to 0} \frac{\cos x \, \frac{\sin x}{\cos x}}{x} = \lim_{x\to 0} \frac{\sin x}{x} = 1$$

1b

$$\lim_{x\to\infty} \left(\sqrt{x^2-4x+100} - x\right) = [\infty - \infty] = \lim_{x\to\infty} \frac{\left(\sqrt{x^2-4x+100}-x\right)\left(\sqrt{x^2-4x+100}+x\right)}{\sqrt{x^2-4x+100}+x}$$

$$= \lim_{x\to\infty} \frac{x^2-4x+100-x^2}{\sqrt{x^2\left(1-\frac{4}{x}+\frac{100}{x^2}\right)}+x} = \lim_{x\to\infty} \frac{x\left(-4+\frac{100}{x}\right)}{|x|\cdot\sqrt{1-\frac{4}{x}+\frac{100}{x^2}}+x}$$

$|x| = x$, då $x>0$

$$= \lim_{x\to\infty} \frac{-4+\frac{100}{x}}{\sqrt{1-\frac{4}{x}+\frac{100}{x^2}}+1} = \frac{-4+0}{\sqrt{1-0+0}+1} = \frac{-4}{2} = -2$$

($\frac{100}{x} \to 0$, $\frac{4}{x} \to 0$, $\frac{100}{x^2} \to 0$)

1c

$$\lim_{x\to 3} \frac{x^2-x-6}{x-3} = \left[\frac{0}{0}\right] = \lim_{x\to 3} \frac{(x-3)(x+2)}{(x-3)\cdot 1} = 3+2 = 5$$

1d

$$\lim_{x\to 0+} (\sin x)\cdot \ln x = \lim_{x\to 0+} \frac{\sin x}{x}\, x\cdot \ln x = 1\cdot 0 = 0$$

($\frac{\sin x}{x} \to 1$, $x \cdot \ln x \to 0$)

$\alpha > 0$: $\lim_{x\to 0+} x^{\alpha}\cdot \ln x = 0$ ($x^{\alpha} \to 0$, $\ln x \to -\infty$)

2a

Tankegång: $\frac{1}{\cos^2 x} = 1 + \tan^2 x \Leftrightarrow \cos^2 x = \frac{1}{1+\tan^2 x}$

Givet: $\tan v = \frac{1}{4}$

$$\sin 2v = 2 \sin v \cos v = 2 \cdot \frac{\sin v}{\cos v} \cdot \cos v \cdot \cos v$$

$$= 2 \cdot \tan v \cdot \cos^2 v = 2 \cdot \tan v \cdot \frac{1}{1+\tan^2 v} = 2 \cdot \frac{1}{4} \cdot \frac{1}{1+\left(\frac{1}{4}\right)^2}$$

$$= \frac{\frac{1}{2}}{1+\frac{1}{16}} = \frac{\frac{1}{2}\cdot 16}{16+1} = \frac{8}{17}$$

$$\cos 2v = 2\cdot\cos^2 v - 1 = 2\frac{1}{1+\tan^2 v} - 1 = 2\frac{1}{1+\left(\frac{1}{4}\right)^2} - 1$$

$$= \frac{2}{1+\frac{1}{16}} - 1 = \frac{2\cdot 16}{16+1} - 1 = \frac{32}{17} - 1 = \frac{32}{17} - \frac{17}{17} = \frac{15}{17}$$

2b

$$\tan v = \frac{1}{4} \qquad v = \arctan\frac{1}{4}$$

$$\tan\left(2\arctan\frac{1}{4}\right) = \tan(2v) = \frac{\sin 2v}{\cos 2v} = \frac{8/17}{15/17} = \frac{8}{15}$$

2c

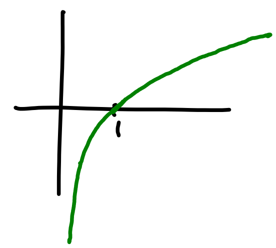
$$f(x) = \cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$$

$$f'(x) = \frac{1}{2}\left(e^x + e^{-x}\cdot(-1)\right) = \frac{1}{2}\left(e^x - e^{-x}\right) \ (= \sinh x)$$

3a

$$\ln 2 + \ln(x^2+1) < \ln(x+1) + \ln(x+2)$$

gitt $\quad x > -1$



$$\ln\left(2\cdot(x^2+1)\right) < \ln\left((x+1)(x+2)\right)$$

$$2\cdot(x^2+1) < (x+1)(x+2)$$

$$2x^2+2 < x^2+3x+2$$

$$x^2 - 3x < 0$$

$$x(x-3) < 0$$

	-1		0		3	
x		$-$	0	$+$		$+$
$x-3$		$-$		$-$	0	$+$
$x\cdot(x-3)$		$+$	0	$-$	0	$+$

Svar: $0 < x < 3$

3b

$$\sqrt{2x^2-1} + 3 = 2x \quad (*)$$

$$\sqrt{2x^2-1} = 2x-3$$

Kvadrera

$$2x^2-1 = (2x-3)^2$$

$$2x^2-1 = 4x^2-12x+9$$

$$0 = 2x^2-12x+10$$

$$0 = x^2-6x+5$$

$$x = 3 \pm \sqrt{9-5} = 3 \pm 2$$

$\boxed{x=5}$ eller $\cancel{x=1}$

sätt in i $(*)$

$7+3=2\cdot 5$ $\qquad$ $1+3=2$

$10=10$ $\qquad$ $4=2$

Ok $\qquad$ falsk rot

Svar: $x=5$

3c

$$\frac{x+1}{x+3} < \frac{1}{|x-1|}$$

1

$$\frac{x+1}{x+3} < -\frac{1}{x-1}$$

$$\frac{x+1}{x+3} + \frac{1}{x-1} < 0$$

$$\frac{\overbrace{(x+1)(x-1)}^{x^2-1} + x+3}{(x+3)(x-1)} < 0$$

$$\frac{x^2+x+2}{(x+3)(x-1)} < 0 \qquad x^2+x+2 > 0$$

$x^2+x+2=0$

$x = -\frac{1}{2} \pm \sqrt{\underbrace{\frac{1}{4}-2}_{<0}}$

		-3		1
x^2+x+2	$+$		$+$	
$x+3$	$-$	0	$+$	
$x-1$	$-$		$-$	0
Kvoten	$+$	ej def.	$-$	

$$-3 < x < 1$$

$$\frac{x+1}{x+3} < \frac{1}{x-1}$$

$$\frac{x+1}{x+3} - \frac{1}{x-1}$$

$$\frac{\overbrace{(x+1)(x-1)}^{x^2-1} - (x+3)}{(x+3)(x-1)} < 0$$

$$\frac{x^2-x-4}{(x+3)(x-1)} < 0$$

$x^2-x-4=0$

$x = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \underset{16/4}{4}}$

$= \frac{1}{2} \pm \sqrt{\frac{17}{4}}$

$= \frac{1 \pm \sqrt{17}}{2}$

$$\frac{\left(x - \frac{1+\sqrt{17}}{2}\right)\left(x - \frac{1-\sqrt{17}}{2}\right)}{(x+3)(x-1)} < 0$$

$x = \frac{1+\sqrt{17}}{2} = 2{,}56$

el

$x = \frac{1-\sqrt{17}}{2} = -1{,}56$

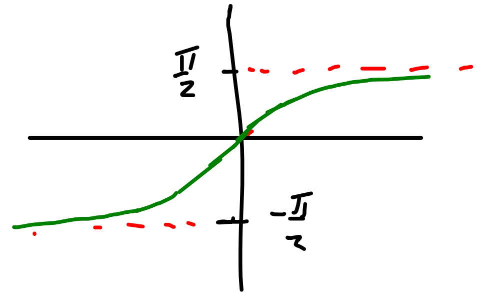
	1		$\frac{1+\sqrt{17}}{2}$	
$x - \frac{1+\sqrt{17}}{2}$		$-$	0	$+$
$x - \frac{1-\sqrt{17}}{2}$		$+$		$+$
$x+3$		$+$		$+$
$x-1$	0	$+$		$+$
Kvoten		$-$	0	$+$

$1 < x < \frac{1+\sqrt{17}}{2}$

SVAR: $-3 < x < 1$ eller $1 < x < \frac{1+\sqrt{17}}{2}$

4a

$$f(x) = x - 2\arctan x$$



Sned asymptot $y = k\cdot x + m$

$$k = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{1}{x}(x - 2\arctan x)$$

$$= \lim_{x\to\infty} \Big(1 - \underbrace{\tfrac{2}{x}}_{\to 0} \underbrace{\arctan x}_{\to \frac{\pi}{2}}\Big) = 1 - 0\cdot\frac{\pi}{2} = 1$$

$$m = \lim_{x\to\infty}(f(x) - k\cdot x) = \lim_{x\to\infty}\Big(\cancel{x} - 2\underbrace{\arctan x}_{\to\frac{\pi}{2}} - \cancel{1\cdot x}\Big) = -\pi$$

dvs. sned asymptot $\boxed{y = 1\cdot x - \pi}$ då $x\to\infty$

$$k = \lim_{x\to-\infty}\frac{f(x)}{x} = \dots = \lim_{x\to-\infty}\Big(1 - \underbrace{\tfrac{2}{x}}_{\to 0}\underbrace{\arctan x}_{\to -\frac{\pi}{2}}\Big) = 1 - 0\cdot(-\tfrac{\pi}{2}) = 1$$

$$m = \lim_{x\to-\infty}(f(x) - k\cdot x) = \lim_{x\to-\infty}\Big(\cancel{x} - 2\underbrace{\arctan x}_{\to -\frac{\pi}{2}} - \cancel{1\cdot x}\Big) = \pi$$

dvs. sned asymptot $\boxed{y = 1\cdot x + \pi}$ då $x\to-\infty$

Inga lodräta asymptoter

- Lokala max/min

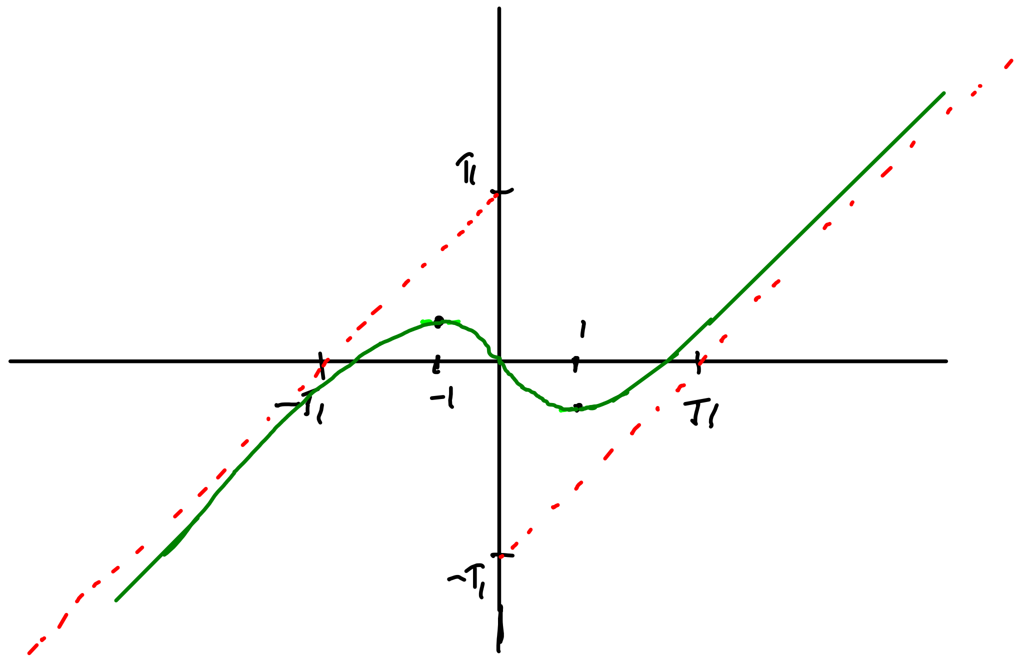
$f(x) = x - 2\arctan x$

$$f'(x) = 1 - 2\,\frac{1}{1+x^2} = \frac{1+x^2-2\cdot 1}{1+x^2} = \frac{x^2-1}{1+x^2} = \frac{(x-1)(x+1)}{1+x^2}$$

x		-1		1	
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	↗	lok. max	↘	lok. min	↗

$f(-1) = -1 + \frac{\pi}{2}$ $\qquad$ $f(1) = 1 - \frac{\pi}{2}$

4b



5a

$$\sinh x = \frac{3}{4}$$

$$\frac{e^x - e^{-x}}{2} = \frac{3}{4}$$

$$e^x - e^{-x} = \frac{3}{2}$$

$$e^x - \frac{1}{e^x} = \frac{3}{2}$$

Variabelbyte: $t = e^x$

$$t - \frac{1}{t} = \frac{3}{2}$$

$$t^2 - 1 = \frac{3}{2}t$$

$$t^2 - \frac{3}{2}t - 1 = 0$$

$$t = \frac{3}{4} \pm \sqrt{\frac{9}{16} + 1} \quad \left(\tfrac{16}{16}\right)$$

$$= \frac{3}{4} \pm \sqrt{\frac{25}{16}}$$

$$= \frac{3}{4} \pm \frac{5}{4}$$

$t = 2$ eller $t = -\frac{1}{2}$

$e^x = 2$

$\ln e^x = \ln 2$

$$\boxed{x = \ln 2}$$

↑ Svar!

$e^x = -\frac{1}{2}$ går ej ty $e^x > 0$

5b

$y = g(x)$ — $g(x)$ är invers till $\sinh(x)$

$$\sinh(y) = x$$

$$\frac{e^y - e^{-y}}{2} = x$$

$$e^y - e^{-y} = 2x$$

Om $x = \frac{3}{4}$ så gäller $\sinh(y) = \frac{3}{4}$ att $y = \ln 2$ enligt 5a.

Derivera m.a.p. x givet $y = y(x)$

$$e^y \cdot y' - e^{-y}(-y') = 2$$

Sätt in $(x,y)=\left(\frac{3}{4}, \ln 2\right)$

$$e^{\ln 2}\cdot y' + e^{-\ln 2}\, y' = 2$$

$$2\cdot y' + \frac{1}{e^{\ln 2}}\cdot y' = 2$$

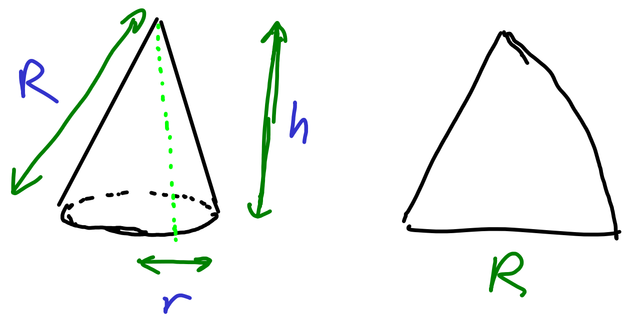
$$\left(2+\frac{1}{2}\right)y' = 2$$

$$(4+1)\,y' = 4$$

$$y' = \frac{4}{5}$$ då $(x,y)=\left(\frac{3}{4}, \ln 2\right)$

Dvs

$$g'\left(\frac{3}{4}\right)=\frac{4}{5}$$



Start: Basradie r

Höjd h

Sidlängd R

Volym V

$$R^2=r^2+h^2$$

$$V=\frac{\pi}{3}r^2\cdot h$$

Om R hålls konstant är $r^2=R^2-h^2$ $\quad 0\le h\le R$

vi får

$$V(h)=\frac{\pi}{3}(R^2-h^2)\cdot h = \frac{\pi}{3}(R^2\cdot h - h^3)$$

$$V'(h)=\frac{\pi}{3}(R^2-3\cdot h^2)$$

$$V'(h)=0 \Leftrightarrow R^2=3\cdot h^2$$

$$h=\pm\frac{R}{\sqrt{3}}$$ stationär punkt

Ändpunkter $h=0,\ h=R$

$V(0)=0$

$V(R)=0$

$$V\left(\frac{R}{\sqrt{3}}\right)=\frac{\pi}{3}\left(R^2-\frac{R^2}{3}\right)\frac{R}{\sqrt{3}}=\frac{\pi}{3}\left(1-\frac{1}{3}\right)\frac{R^3}{\sqrt{3}}=\frac{2\cdot\pi\cdot R^3}{9\cdot\sqrt{3}}$$

(where $1-\frac{1}{3}$ is marked $2/3$)

Svar: $\frac{2\cdot\pi\cdot R^3}{9\cdot\sqrt{3}}$