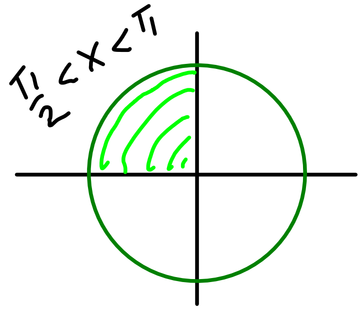
Tentamen 150819

(a) $\quad x^2-6x+9$

$$f(x) = x^2 - 6x + 12 = (x-3)^2 - 9 + 12 = (x-3)^2 + 3$$

Svar: $f(3) = 3$, minsta värde

(b)



$$\cos x = -\frac{2}{3}$$

$$\sin x > 0$$

$$\sin^2 x + \cos^2 x = 1$$

$$\sin x = \pm\sqrt{1-\cos^2 x} = \sqrt{1-\left(-\tfrac{2}{3}\right)^2} = \sqrt{\underset{\frac{9}{9}}{1} - \frac{4}{9}} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\sqrt{5}/3}{-2/3} = -\frac{\sqrt{5}}{2}$$ ← Svar!

(c)

$$\ln(x^2) + \ln(x^3) = 10$$

$$2\ln x + 3\ln x = 10$$

$$5\ln x = 10$$

$$\ln x = 2$$

$$e^{\ln x} = e^2$$

$$\boxed{x = e^2}$$ ← Svar

$\boxed{2}$ (a)

$$\lim_{x\to 2}\left(\frac{x-2}{x-4} - \frac{x+2}{x+4}\right) = \frac{0}{-2} - \frac{4}{6} = -\frac{2}{3}$$

(b)

$$\lim_{x\to\infty}\frac{x^2+x-\sin 3x}{3x^2+\cos x} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty}\frac{\cancel{x^2}\left(1 + \underbrace{\tfrac{1}{x}}_{\to 0} - \underbrace{\tfrac{1}{x^2}}_{\to 0}\underbrace{\sin 3x}_{\le 1,\ \ge -1}\right)}{\cancel{x^2}\left(3 + \underbrace{\tfrac{1}{x^2}}_{\to 0}\underbrace{\cos x}_{\le 1,\ \ge -1}\right)}$$

$$= \frac{1+0-0}{3+0} = \frac{1}{3}$$

2 (c)

$$\lim_{x\to 0}\frac{\sin^2 x}{1-\cos x} = \left[\frac{0}{0}\right] = \lim_{x\to 0}\frac{1-\cos^2 x}{1-\cos x} = \lim_{x\to 0}\frac{\cancel{(1-\cos x)}(1+\cos x)}{\cancel{(1-\cos x)}}$$

$$= 1+1 = 2$$

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3 (a)

$$\frac{1}{x} < \frac{5-2x}{x+2}$$

$$\frac{1}{x} - \frac{5-2x}{x+2} < 0$$

$$\frac{x+2}{x\cdot(x+2)} - \frac{x(5-2x)}{x(x+2)} < 0$$

$$\frac{x+2-5x+2x^2}{x(x+2)} < 0$$

$$\frac{2x^2-4x+2}{x(x+2)} < 0$$

$$\frac{2(x^2-2x+1)}{x(x+2)} < 0$$

$$\frac{2(x-1)^2}{x(x+2)} < 0$$

x		-2		0		1	
$(x-1)^2$	+		+		+	0	+
x	−		−	0	+		+
$x+2$	−	0	+		+		+
$\frac{2(x-1)^2}{x(x+2)}$	+	ej def.	−	ej def.	+	0	+

Svar: $-2 < x < 0$

3 (b) $\quad |x-1| < 2|x-2|$

$$-(x-1) < -2(x-2)$$
$$-x+1 < -2x+4$$
$$x < 3$$
sant för alla
$$\boxed{x \leq 1}$$

$$x-1 < -2(x-2)$$
$$x-1 < -2x+4$$
$$3x < 5$$
$$x < \frac{5}{3}$$
sann för alla
$$\boxed{1 < x < \frac{5}{3}}$$

$$x-1 < 2(x-2)$$
$$x-1 < 2x-4$$
$$\boxed{3 < x}$$
sant utan restriktion

Svar: $\quad x < \frac{5}{3} \quad$ eller $\quad x > 3$

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4 (a) $\quad \tan\left(\frac{y}{2}\right) = \frac{2}{\pi}\cdot x \cdot y \quad , \quad (x,y) = \left(1, \frac{\pi}{2}\right)$

Derivera map x, givet att $y = y(x)$

$$\frac{1}{\cos^2\left(\frac{y}{2}\right)}\cdot\frac{1}{2}\cdot y' = \frac{2}{\pi}\left(1\cdot y + x\cdot y'\right)$$

$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x} = 1+\tan^2 x$

sätt in $(x,y) = \left(1, \frac{\pi}{2}\right)$

$$\frac{1}{\cos^2\frac{\pi}{4}}\cdot\frac{1}{2}\cdot y' = \frac{2}{\pi}\left(1\cdot\frac{\pi}{2} + 1\cdot y'\right)$$

$$\underbrace{\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2}\,\frac{1}{2}}_{=1}\cdot y' = 1 + \frac{2}{\pi}y'$$

$$1\cdot y' - \frac{2}{\pi}\cdot y' = 1$$

$$\left(1-\frac{2}{\pi}\right)y' = 1$$

$$y' = \frac{1}{1-\frac{2}{\pi}} = \frac{\pi}{\pi-2}$$

Svar: $\quad y'(1) = \frac{\pi}{\pi-2}$

4 (b) Givet: $f(x) = \dfrac{x}{\cos x}$ Sökes: $(f^{-1})'(0)$

$$y = f^{-1}(x)$$

$$f(y) = x$$

$$\frac{y}{\cos y} = x \qquad x = 0 \quad y = 0$$

$$y = x \cdot \cos y$$

Derivera map. x givet $y = y(x)$

$$y' = 1 \cdot \cos y + x(-\sin y) \cdot y'$$

sätt in $(x, y) = (0, 0)$

$$y' = 1 \cdot \cos 0 + 0 \cdot (-\sin 0) \cdot y'$$

$$y' = 1$$

<u>Svar:</u> $(f^{-1})'(0) = 1$

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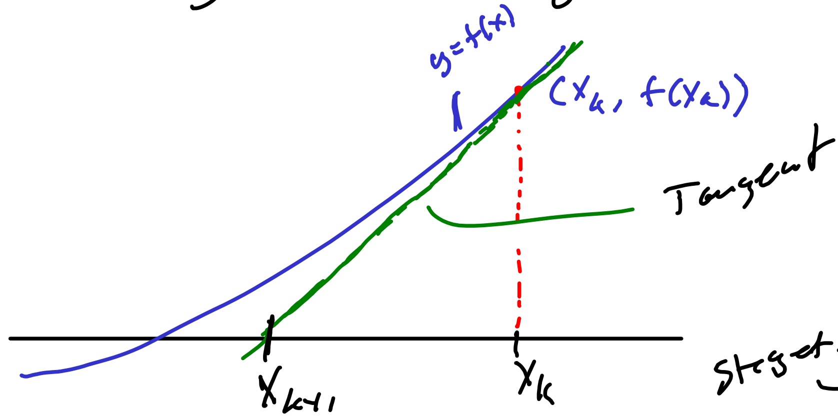
5 (a)

Newton Raphson finner en approximation till x^* som löser $f(x) = 0$, genom att bilda en talföljd $x_0, x_1, x_2, x_3, \ldots$ som uppfyller $\lim_{k \to \infty} x_k = x^*$. Talföljden bildas av

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Motivering till Newtonsteget:



steget $\rightarrow$

$$y - f(x_k) = f'(x_k)(x - x_k)$$

välj x så $y = 0$

$$0 - f(x_k) = f'(x_k)(x - x_k)$$

$$-\frac{f(x_k)}{f'(x_k)} = x - x_k$$

$$x = x_k - \frac{f(x_k)}{f'(x_k)}$$

$\boxed{5}$ (b)

$$e^{-x^2} = \arctan \frac{x}{4}$$

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$

$$e^{-x^2} - \arctan \frac{x}{4} = 0$$

$$f(x) = e^{-x^2} - \arctan \frac{x}{4}$$

$$f'(x) = e^{-x^2}(-2x) - \frac{1}{1+\left(\frac{x}{4}\right)^2} \cdot \frac{1}{4}$$

$$= -2xe^{-x^2} - \frac{1/4}{1+\frac{x^2}{16}} = -2xe^{-x^2} - \frac{4}{16+x^2}$$

Newtonsteget

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{e^{-x_k^2} - \arctan\left(\frac{x_k}{4}\right)}{-2x_k e^{-x_k^2} - \frac{4}{16+x_k^2}}$$

$x_0 = 0$
$x_1 = 4$
$x_2 = -2{,}2831$
$x_3 = 0{,}9168$
$x_4 = 1{,}1174$
$x_5 = 1{,}1340$

$x_0 = 1$
$x_1 = 1{,}1266$
$x_2 = 1{,}1341$
$x_3 = 1{,}1342$
$x_4 = 1{,}1342$
$x_5 = 1{,}1342$

$\boxed{6}$ $y = 1 + x^{3/2}, \qquad x \geq 0$

Kortaste avstånd till (8.1), för (x,y) ?

$$A(x) = \sqrt{(x-8)^2 + (y-1)^2} = \sqrt{(x-8)^2 + (\cancel{1} + x^{3/2} - \cancel{1})^2}$$

$$= \sqrt{x^2 - 16x + 64 + x^3} = \sqrt{x^3 + x^2 - 16x + 64}$$

Minimera avståndet i kvadrat är lättare

$$s(x) = (A(x))^2 = x^3 + x^2 - 16x + 64$$

$$s'(x) = 3x^2 + 2x - 16 = 3\left(x^2 + \frac{2}{3}x - \frac{16}{3}\right)$$

$$x^2 + \frac{2}{3}x - \frac{16}{3} = 0$$

$$x = -\frac{1}{3} \pm \sqrt{\frac{1}{9} + \frac{16}{3}} = -\frac{1}{3} \pm \sqrt{\frac{1}{9} + \frac{48}{9}} = -\frac{1}{3} \pm \sqrt{\frac{49}{9}} = -\frac{1}{3} \pm \frac{7}{3}$$

$x = \frac{6}{3} = 2$ eller $x = -\frac{8}{3}$

$$s'(x) = 3 \cdot (x-2)\left(x + \frac{8}{3}\right)$$

x	0		2	
$x-2$		$-$	0	$+$
$x+\frac{8}{3}$		$+$		$+$
$f'(x)$		$-$	0	$+$
$f(x)$		$\searrow$	minsta värde	$\nearrow$

Minst avstånd då $x = 2$

$$y = 1 + 2^{3/2} = 1 + 2\sqrt{2}$$