Lektionsuppgift 2

$$z^2-(3+i)z+(4+3i)=0$$

kvadratkomplettera

$$\left(z-\frac{3+i}{2}\right)^2-\left(\frac{3+i}{2}\right)^2+(4+3i)=0$$

$$\left(z-\frac{3+i}{2}\right)^2=\frac{9+6i+\overbrace{i^2}^{=-1}}{4}-4-3i$$

$$\left(z-\frac{3+i}{2}\right)^2=2+\frac{3}{2}i-4-3i$$

$$\left(z-\frac{3+i}{2}\right)^2=-2-\frac{3}{2}i$$

variabelbyte

$$w=z-\frac{3+i}{2} \qquad z=w+\frac{3}{2}+i\frac{1}{2}$$

med

$$w=a+ib \qquad w^2=a^2+i2ab+i^2b^2=a^2-b^2+i2ab$$

ger

$$w^2=-2-\frac{3}{2}i$$

$$a^2-b^2+i2ab=-2-\frac{3}{2}i$$

dvs

$$(*)\quad \begin{cases} 1.\ a^2-b^2=-2 \\ 2.\ 2ab=-\frac{3}{2} \end{cases}$$

2. ger $b=-\frac{3}{4a}$ i 1. ger

$$a^2-\frac{9}{16a^2}=-2$$

$$(a^2)^2+2a^2-\frac{9}{16}=0$$

$$a^2=-1\pm\sqrt{1+\frac{9}{16}}$$

$$=-1\pm\sqrt{\frac{16}{16}+\frac{9}{16}}$$

$$=-1\,(\pm)\,\frac{5}{4}=\frac{1}{4}$$

dvs

$a=\frac{1}{2}$ eller $a=-\frac{1}{2}$

$b=-\frac{3}{4a}$: $b=-\frac{3}{2}$ $\qquad b=\frac{3}{2}$

$w=\frac{1}{2}-\frac{3}{2}i \qquad w=-\frac{1}{2}+\frac{3}{2}i$

$z=w+\frac{3}{2}+\frac{1}{2}i$: $\boxed{z=2-i \qquad z=1+2i}$

Lektionsuppgift 2, alternativ avslutning

$$w^2 = -2 - \frac{3}{2}i \quad \text{ger}$$

$$|w|^2 = \sqrt{(-2)^2 + \left(-\frac{3}{2}\right)^2} \quad \text{dvs}$$

$$a^2 + b^2 = \sqrt{\frac{25}{4}} = \frac{5}{2}$$

använd som extra villkor i ekv. syst (*)

$$\begin{cases} a^2 + b^2 = \frac{5}{2} & (1) \\ a^2 - b^2 = -2 & (2) \\ 2ab = -\frac{3}{2} & (3) \end{cases}$$

1 + 2 ger

$$2a^2 = \frac{5}{2} - 2 = \frac{1}{2}$$

$$a^2 = \frac{1}{4}$$

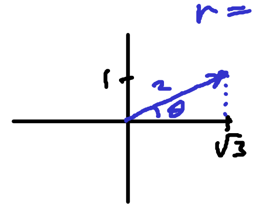
$a = \frac{1}{2}$ eller $a = -\frac{1}{2}$

3 ger $b = -\frac{3}{4a} \Rightarrow b = -\frac{3}{2}$; $b = \frac{3}{2}$

$w = \frac{1}{2} - \frac{3}{2}i$; $w = -\frac{1}{2} + \frac{3}{2}i$

$z = w + \frac{3}{2} + \frac{1}{2}i$; $z = 2 - i$; $z = 1 + 2i$

Lektionsuppgift 3



$r = \sqrt{1^2 + (\sqrt{3})^2} = 2$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

$$z = \sqrt{3} + i$$

så

$$z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$z^3 = 2^3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^3$$

$$= 8\left(\underbrace{\cos\frac{3\pi}{6}}_{=0} + i\underbrace{\sin\frac{3\pi}{6}}_{=1}\right)$$

$$= 8i$$

alt.

$$z = 2e^{i\frac{\pi}{6}}$$

$$z^3 = 2^3\left(e^{i\frac{\pi}{6}}\right)^3$$

$$= 8\,e^{i\frac{3\pi}{6}}$$

$$= 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= 8i$$

Lektionsuppgift 4

$f(x) = 0 \iff z^6 = 1 \quad (*)$

Polär form

$z = r \cdot e^{i\theta}$ $\qquad 1 = 1 \cdot e^{i n \cdot 2\pi}$ $\qquad n \in \mathbb{Z}$

$z^6 = r^6 \cdot e^{i6\theta}$

i $(*)$ ger

$$r^6 \cdot e^{i6\theta} = 1 \cdot e^{in2\pi}$$

dvs $\begin{cases} r^6 = 1 \\ 6\theta = n2\pi \end{cases}$ $\qquad \begin{matrix} r = 1 \\ \theta = n \cdot \frac{\pi}{3} \end{matrix}$

ger lösningar

$$z_n = 1 \cdot e^{in\frac{\pi}{3}} = \cos n\tfrac{\pi}{3} + i \sin n\tfrac{\pi}{3}$$

$n = 0 \quad z_0 = \cos 0 + i \sin 0 = 1$

$n = 1 \quad z_1 = \cos\frac{\pi}{3} + i \sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$n = 2 \quad z_2 = \cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$n = 3 \quad z_3 = \cos\pi + i \sin\pi = -1$

$n = 4 \quad z_4 = \cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$n = 5 \quad z_5 = \cos\frac{5\pi}{3} + i \sin\frac{5\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

ger talkforer

$z - z_0 = z - 1$

$(z - z_1)(z - z_5) = (z - \frac{1}{2} - i\frac{\sqrt{3}}{2})(z - \frac{1}{2} + i\frac{\sqrt{3}}{2}) = z^2 - z + 1$

$(z - z_2)(z - z_4) = (z + \frac{1}{2} - i\frac{\sqrt{3}}{2})(z + \frac{1}{2} + i\frac{\sqrt{3}}{2}) = z^2 + z + 1$

$(z - z_3) = z + 1$

dvs

$$f(z) = z^6 - 1 = (z-1)(z+1)(z^2 - z + 1)(z^2 + z + 1)$$

Lektionsuppgift 6

$$x^2y'' - xy' + y = 0$$

1. Ordningstalet är 2

2\. i $\left.\begin{array}{l} y = x \\ y' = 1 \\ y'' = 0 \end{array}\right\}$ i V.L. ger $x^2 \cdot \underbrace{0}_{y''} - x \cdot \underbrace{1}_{y'} + \underbrace{x}_{y} = 0$

för alla x, dvs lösning

ii $\left.\begin{array}{l} y = x^2 \\ y' = 2x \\ y'' = 2 \end{array}\right\}$ i V.L ger $x^2 \cdot \underbrace{2}_{y''} - x \cdot \underbrace{2x}_{y'} + \underbrace{x^2}_{y} = x^2$

ej $= 0$ i allmänhet. Ingen lösning.

iii $\left.\begin{array}{l} y = x\ln x \\ y' = \ln x + 1 \\ y'' = \frac{1}{x} \end{array}\right\}$ i V.L ger $x^2 \cdot \underbrace{\frac{1}{x}}_{y''} - x\underbrace{(\ln x + 1)}_{y'} + \underbrace{x\ln x}_{y}$

$= x - x\ln x - x + x\ln x = 0$

för alla x, dvs lösning

iv $\left.\begin{array}{l} y = x^2\ln x \\ y' = 2x\ln x + x \\ y'' = 2\ln x + 3 \end{array}\right\}$ i V.L ger $x^2(2\ln x + 3) - x(2x\ln x + x) + x^2\ln x$

$= 2x^2\ln x + 3x^2 - 2x^2\ln x - x^2 + x^2\ln x$

$= 2x^2 + x^2\ln x$

ej 0 för alla x, dvs ingen lösning

3. Partikulärlösningar

4. Nej! Riktningsfält kan endast göras för första ordningens differentialekvationer.

Lektionsuppgift 7

$$\frac{1}{\cos x}y' - y = 1$$

$$y' + (-\cos x)y = \cos x$$

primitiv fkn till $-\cos x$ är $-\sin x$

ger i.f. $e^{-\sin x}$, så

$$y'e^{-\sin x} + (-\cos x)ye^{-\sin x} = \cos x\, e^{-\sin x}$$

$$\frac{d}{dx}\left(y e^{-\sin x}\right) = \cos x e^{-\sin x}$$

$$y e^{-\sin x} = \int \cos x\, e^{-\sin x}\, dx = \left[\begin{array}{l} u = -\sin x \\ \frac{du}{dx} = -\cos x \\ -du = \cos x\, dx \end{array}\right]$$

$$= \int e^u(-du) = -e^u + C$$

$$= -e^{-\sin x} + C$$

mult. m $e^{\sin x}$

$$\boxed{\begin{aligned} y &= -1 + Ce^{\sin x} \\ &= Ce^{\sin x} - 1 \end{aligned}}$$

Villkoret $y(0) = 0$ ger

$$0 = C \cdot \underbrace{e^0}_{=1} - 1 \Leftrightarrow C = 1$$

dvs

$$\boxed{y = e^{\sin x} - 1}$$

Lektionsuppgift 8

$$y' - xy^2 = x$$

$$y' = x + xy^2$$

$$y' = x(1+y^2)$$

$$\underbrace{\frac{1}{1+y^2}}_{\text{bara "y"}} y' = \underbrace{x}_{\text{bara "x"}}$$

separabel

$$\frac{1}{1+y^2} \frac{dy}{dx} = x$$

$$\int \frac{1}{1+y^2}\, dy = \int x\, dx$$

$$\arctan y = \underbrace{\frac{x^2}{2} + C}_{\in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[}$$

$$\boxed{y = \tan\left(\frac{x^2}{2} + C\right)}$$

Lektionsuppgift 9

$$y'' + 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

1. Allmän lösning

Karakteristisk ekv. $r^2 + 2r + 1 = 0$

$$r = -1 \pm \sqrt{1-1} = -1, \quad r_1 = r_2 = -1$$

ger $y = (Ax + B)e^{-x}$

2. Partikulärlösning för begynnelsevillkoren

Vi behöver

$$y(x) = (Ax + B)e^{-x}$$

$$y'(x) = Ae^{-x} + (Ax + B)e^{-x}(-1)$$

$$= (-Ax + A - B)e^{-x}$$

så

$$1 = y(0) = (A \cdot 0 + B)e^{-0} = B, \quad \text{dvs } B = 1$$

$$0 = y'(0) = (-A \cdot 0 + A - 1)e^{-0} = A - 1, \quad \text{dvs } A = 1$$

ger $y = (1 \cdot x + 1)e^{-x} = (x+1)e^{-x}$

Lektionsuppgift 10

$$y'' - 4y' + 4y = x^2 + 1$$

Ansats: $y_p = ax^2 + bx + c$

$y_p' = 2ax + b$

$y_p'' = 2a$

i diff. ekv. ger

$$\underbrace{2a}_{y_p''} - 4\underbrace{(2ax + b)}_{y_p'} + 4\underbrace{(ax^2 + bx + c)}_{y_p} = x^2 + 1$$

$$4ax^2 + (-8a + 4b)x + (2a - 4b + 4c) = x^2 + 1$$

Polynomlikhet ger

$$\begin{array}{l} x^2 \\ x^1 \\ x^0 \end{array} \begin{cases} 4a = 1 \\ -8a + 4b = 0 \\ 2a - 4b + 4c = 1 \end{cases}$$

$a = \frac{1}{4}$

$b = \frac{2}{4} = \frac{1}{2}$

$c = \frac{1}{4}(1 - \frac{1}{2} + 2) = \frac{1}{4} \cdot \frac{5}{2} = \frac{5}{8}$

ger partikulärlösning

$$y_p = \overbrace{\frac{1}{4}}^{a}x^2 + \overbrace{\frac{1}{2}}^{b}x + \overbrace{\frac{5}{8}}^{c}$$

Lektionsuppgift 11

$$y'' + 3y' + 2y = xe^{-x}, \quad y(0) = 1, \quad y'(0) = 0$$

1. Homogenlösning till $y'' + 3y' + 2y = 0$

Karakteristisk ekv: $r^2 + 3r + 2 = 0$

$$r = -\frac{3}{2} \pm \sqrt{\frac{9}{4} - \underset{=\frac{8}{4}}{2}} = -\frac{3}{2} \pm \frac{1}{2}$$

$r_1 = -2 \qquad r_2 = -1$

$$\boxed{y_h = Ae^{-2x} + Be^{-x}}$$

Uppg. 11 forts

2. Partikulärlösning för högerledet $\quad y''+3y'+2y = xe^{-x}$

Ansats: $y_p = z\cdot e^{-x}$

$$y_p' = z'e^{-x} + z e^{-x}(-1) = e^{-x}(z'-z)$$

$$y_p'' = e^{-x}(-1)(z'-z) + e^{-x}(z''-z') = e^{-x}(z''-2z'+z)$$

i diff. ekv. ger

$$\cancel{e^{-x}}(z''-2z'+z)+3\cancel{e^{-x}}(z'-z)+2\cdot z\cdot\cancel{e^{-x}} = x\cancel{e^{-x}}$$

$$z''+z' = x$$

Ansats $\quad z = x(a\cdot x+b) = a\cdot x^2+b\cdot x$

$$z' = 2ax+b$$

$$z'' = 2a$$

i diff. ekv. ger

$$2a + 2ax+b = x$$

$$2a\,x + (2a+b) = x$$

$$\begin{array}{l} (x) \\ (1) \end{array}\begin{cases} 2a = 1 \\ 2a + b = 0 \end{cases}$$

$$a = \frac{1}{2} \qquad b = -1$$

dvs

$$z = \frac{1}{2}x^2 - x$$

$$y_p = z e^{-x} = \left(\frac{1}{2}x^2 - x\right)e^{-x}$$

3. Allmän lösning

$$y = y_h + y_p = Ae^{-2x} + Be^{-x} + \left(\frac{1}{2}x^2-x\right)e^{-x}$$

4. Begynnelsevillkor $\quad y(0)=1 \qquad y'(0)=0$

$$y' = -2Ae^{-2x} - Be^{-x} + (x-1)e^{-x} - \left(\frac{1}{2}x^2-x\right)e^{-x}$$

$$= -2Ae^{-2x} - Be^{-x} + \left(-\frac{1}{2}x^2 - 1\right)e^{-x}$$

$$1 = y(0) = A\cdot 1 + B\cdot 1$$

$$0 = y'(0) = -2A - B - 1$$

$$\Rightarrow \begin{array}{l} (1) \\ (2) \end{array}\begin{cases} A + B = 1 \\ 2A + B = -1 \end{cases}$$

$$(2)-(1) \qquad A = -2$$

$$B = 3$$

Svar: $\quad y = -2e^{-2x} + 3e^{-x} + \left(\frac{1}{2}x^2 - x\right)e^{-x}$

Lektionsuppgift 13

$$\frac{1}{x^2+x+3} < \frac{1}{x^2}$$

$$0 < \int_1^\infty \underbrace{\frac{1}{x^2+x+3}}_{>0}\,dx < \int_1^\infty \frac{1}{x^2}\,dx$$

Konvergent $\Longleftarrow$ Konvergent ty $p=2>1$

Jämförelsekriteriet

Lektionsuppgift 14

$$S_n = \sum_{k=1}^{n} k\cdot r^k = \frac{r\left(n\cdot r^{n+1} - (n+1)r^n + 1\right)}{(r-1)^2} = \frac{r^{n+1}\left(r-(n+1)\right)+r}{(r-1)^2}$$

tillväxtegenskapen ger

$$= \frac{\underbrace{r^{n+1}}_{\to 0,\ |r|<1;\ \to\infty,\ |r|>1}\underbrace{(-n+r-1)}_{\to -\infty}+r}{(r-1)^2} \to \frac{0+r}{(r-1)^2} \quad \text{då } |r|<1 \text{ och } n\to\infty$$

dvs konvergent då $|r|<1$

och

$$\sum_{k=1}^{\infty} k r^k = \frac{r}{(r-1)^2}, \quad \text{då } |r|<1$$

Lektionsuppgift 15

1. $$0 < \sum_{k=1}^{\infty} \underbrace{\frac{k-1}{k^3+k}}_{\geq 0} < \sum_{k=1}^{\infty} \frac{k}{k^3} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Konvergent $\Longleftarrow$ Konvergent ty $p=2>1$

2. $$\sum_{k=1}^{\infty} -\underbrace{\frac{\ln\left(\frac{1}{k}\right)}{k^2}}_{=0 \text{ då } k=1} = \sum_{k=2}^{\infty} -\frac{\ln\left(\frac{1}{k}\right)}{k^2}, \qquad a_k = -\frac{\ln\left(\frac{1}{k}\right)}{k^2} = f(k)$$

undersök $f(x) = -\frac{\ln\frac{1}{x}}{x^2}$

$$f'(x) = -\frac{\frac{1}{1/x}\cdot\frac{-1}{x^2}\,x^2 - \ln\frac{1}{x}\,2x}{x^4} = -\frac{-x\left(1+2\ln\frac{1}{x}\right)}{x^4} = \frac{1-2\overbrace{\ln x}^{>0{,}5,\ x\geq 2}}{x^3} < 0 \text{ då } x\geq 2$$

Cauchys integralkriterium

$$\int_2^\infty f(x)\,dx = \lim_{X\to\infty} \int_2^X -\frac{\ln\left(\frac{1}{x}\right)}{x^2}\,dx = \left[\begin{array}{l|l} u=\frac{1}{x} & x=2 \\ & u=1/2 \\ \frac{du}{dx}=-\frac{1}{x^2} & x=X \\ du=-\frac{1}{x^2}dx & u=\frac{1}{X}\end{array}\right] = \lim_{X\to\infty} \int_{1/2}^{1/X} 1\cdot\ln(u)\,du$$

$$= \lim_{U\to 0}\left(\left[u\ln u\right]_{1/2}^{U} - \int_{1/2}^{U} u\underbrace{\frac{1}{u}}_{=1}\,du\right) = \lim_{U\to 0}\left(U\ln U - \frac{1}{2}\ln\frac{1}{2} - \left[u\right]_{1/2}^{U}\right)$$

$$= \lim_{U\to 0}\left(\underbrace{U\ln U}_{\to 0} - \frac{1}{2}\ln\frac{1}{2} - \underbrace{U}_{\to 0} + \frac{1}{2}\right) = -\frac{1}{2}\ln\frac{1}{2} + \frac{1}{2} \quad \text{dvs Konvergent}$$

$\Leftarrow$ Serien också konvergent

3. $$\sum_{k=1}^{\infty} \frac{k^3}{3^k} \qquad a_k = \frac{k^3}{3^k}$$

$$\rho = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{(k+1)^3}{3^{k+1}}}{\frac{k^3}{3^k}} = \lim_{k\to\infty} \frac{(k+1)^3}{k^3}\cdot\frac{3^k}{\underbrace{3^{k+1}}_{3^k\cdot 3^1}} = \lim_{k\to\infty}\frac{\cancel{k^3}\left(1+\underbrace{\tfrac{1}{k}}_{\to 0}\right)^3\cdot 1}{\cancel{k^3}\cdot 3}$$

$$= \frac{1}{3} < 1 \quad \text{dvs konvergent}$$

Lektionsuppgift 16

(a) $$a_k = \frac{3}{k}\cdot\left(\frac{2}{5}\right)^k$$

$$L = \lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\frac{\frac{\cancel{3}^1}{k+1}\underbrace{\left(\frac{2}{5}\right)^{k+1}}_{\left(\frac{2}{5}\right)^k\cdot\frac{2}{5}}}{\frac{\cancel{3}^1}{k}\cdot\left(\frac{2}{5}\right)^k} = \lim_{k\to\infty}\frac{k\cdot\frac{2}{5}}{k+1}$$

$$= \lim_{k\to\infty}\frac{2}{5}\,\frac{\cancel{k}^1}{\cancel{k}^1\left(1+\underbrace{\tfrac{1}{k}}_{\to 0}\right)} = \frac{2}{5}\cdot\frac{1}{1\cdot(1+0)} = \frac{2}{5}$$

Konvergensradie $R = \frac{1}{L} = \frac{1}{2/5} = \frac{5}{2}$

dvs konvergent om $-\frac{5}{2} < x < \frac{5}{2}$

(b)

$$f(x) = e^{-x}\cos x + 1$$
$$f'(x) = e^{-x}(-1)\cos x + e^{-x}(-\sin x)$$
$$= e^{-x}(-\cos x - \sin x)$$
$$f''(x) = e^{-x}(-1)(-\cos x - \sin x) + e^{-x}(\sin x - \cos x)$$
$$= e^{-x}\, 2\sin x$$
$$f'''(x) = e^{-x}(-1)\, 2\sin x + e^{-x}\, 2\cos x$$
$$= e^{-x}2(-\sin x + \cos x)$$
$$f^{(4)}(x) = e^{-x}(-1)2(-\sin x + \cos x) + e^{-x}2(-\cos x - \sin x)$$
$$= -e^{-x}2^2\cos x$$
$$f^{(5)}(x) = -e^{-x}(-1)2^2\cos x - e^{-x}2^2(-\sin x)$$
$$= e^{-x}2^2(\cos x + \sin x)$$

$f(0) = 1 + 1 = 2$

$f'(0) = -1$

$f''(0) = 0$

$f'''(0) = 2$

$f^{(4)}(0) = -2^2$

$f^{(5)}(0) = 2^2$

Mönster

$f^{(6)}(0) = 0 \quad f^{(7)}(0) = -2^2 \quad f^{(8)}(0) = 2^3 \quad f^{(9)}(0) = -2^3, 0, 2^3, -2^4, 2^4, 0$

så

$$y(x) = 1 + \sum_{k=0}^{\infty}(-1)^k \frac{2^{2k}}{(4k)!}x^{4k} + \sum_{k=0}^{\infty}(-1)^{k+1}\frac{2^{2k}}{(4k+1)!}x^{4k+1} + \sum_{k=0}^{\infty}(-1)^k\frac{2^{2k+1}}{(4k+3)!}x^{4k+3}$$

Lektionsuppgift 17

$$\lim_{x\to 0}\frac{\cos x - 1}{\ln(1+x) - \sin x} = \lim_{x\to 0}\frac{1 - \frac{x^2}{2!} + O(x^4) - 1}{x - \frac{x^2}{2} + O(x^3) - \left(x - \frac{x^3}{3!} + O(x^5)\right)}$$

$$= \lim_{x\to 0}\frac{-\frac{1}{2}x^2 + O(x^4)}{-\frac{x^2}{2} + \frac{x^3}{6} + O(x^3) - O(x^5)} = \text{[dela med } x^2 \text{ i täljare \& nämnare]}$$

$$= \lim_{x\to 0}\frac{-\frac{1}{2} + O(x^2)}{-\frac{1}{2} + \frac{x}{6} + O(x^3) - O(x^5)} = \frac{-\frac{1}{2} + 0}{-\frac{1}{2} + 0 + 0 - 0} = 1$$

Lektionsuppgift 18

Potensserielösning till $y' - y = 0$

$$y = \sum_{k=0}^{\infty} c_k \cdot x^k$$

$$y' = \sum_{k=1}^{\infty} c_k \cdot k \cdot x^{k-1}$$

ger

$$\sum_{k=1}^{\infty} c_k \cdot k \cdot x^{k-1} - \sum_{k=0}^{\infty} c_k \cdot x^k = 0$$

$$\Big\downarrow k \to k+1$$

$$\sum_{k=0}^{\infty} c_{k+1} (k+1) x^k - \sum_{k=0}^{\infty} c_k \cdot x^k = 0$$

$$\sum_{k=0}^{\infty} \left(c_{k+1}(k+1) - c_k \right) x^k = 0$$

identiskt lika med noll endast då alla koefficienter är noll

$$c_{k+1}(k+1) - c_k = 0$$

$$c_{k+1} = \frac{1}{k+1} c_k$$

ger

$$c_1 = \frac{1}{1} \cdot c_0$$

$$c_2 = \frac{1}{2} \cdot c_1 = \frac{1}{2 \cdot 1} \cdot c_0$$

$$c_3 = \frac{1}{3} c_2 = \frac{1}{3 \cdot 2 \cdot 1} \cdot c_0$$

$$c_4 = \frac{1}{4} c_3 = \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} c_0$$

$$\vdots$$

$$c_k = \frac{1}{k} c_{k-1} = \frac{1}{k \cdot (k-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1} c_0 = \frac{1}{k!} c_0$$

så

$$y = \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} c_0 \cdot \frac{1}{k!} x^k = c_0 \cdot \underbrace{\sum_{k=0}^{\infty} \frac{x^k}{k!}}_{\text{Maclaurinserie av } e^x}$$

Lektionsuppgift 19

$$f(t) = 1 + 2t - t^2 + \sin t - 2\cos t$$

$$F(s) = \mathcal{L}\{f(t)\} = \frac{1}{s} + 2\frac{1}{s^2} - \frac{2}{s^3} + \frac{1}{s^2+1} - 2\frac{s}{s^2+1}$$

$$= \frac{1}{s} + \frac{2}{s^2} - \frac{2}{s^3} + \frac{1-2s}{s^2+1}$$

Lektionsuppgift 20

$$F(s) = \frac{4s+4}{s^2(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2}$$

$$= \frac{A\overbrace{s(s+2)}^{s^2+2s}}{s^2(s+2)} + \frac{B(s+2)}{s^2(s+2)} + \frac{Cs^2}{s^2(s+2)}$$

$$= \frac{(A+C)s^2 + (2A+B)s + 2B}{s^2(s+2)}$$

$$\begin{array}{ll} s^2 & \\ s & \\ 1 & \end{array} \begin{cases} A + C = 0 \\ 2A + B = 4 \\ 2B = 4 \end{cases} \qquad \begin{aligned} B &= 2 \\ A &= 1 \\ C &= -1 \end{aligned}$$

så

$$Y = \frac{1}{s} + 2\frac{1}{s^2} - \frac{1}{s+2}$$

inverstransform ger

$$y = \mathcal{L}^{-1}\{Y\} = 1 + 2t - e^{-2t}$$

Lektionsuppgift 21 a

$$f(t) = e^{-3t}t^2 \qquad \mathcal{L}\{t^2\} = \frac{2}{s^3}$$

$s \to s+3$ av $e^{-3t}\cdot\ldots$

$$F(s) = \mathcal{L}\{f(t)\} = \frac{2}{(s+3)^3}$$

Lektionsuppgift 21 b

$$F(s) = \frac{s+2}{s^2+2s+2} = \frac{s+2}{(s+1)^2-1+2} = \frac{(s+1)+1}{(s+1)^2+1}$$

$$= \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$$

$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

$$f(t) = e^{-t}\cdot\cos t + e^{-t}\cdot\sin t$$

Lektionsuppgift 22

1. Faktorisera $s^3+6s^2+12s+8$

Testar $s=$: $\pm 1 \quad \pm 2 \quad \pm 4 \quad \pm 8$ ← alla tal som delar 8

Valet $s=-2$ är nollställe. Dela med $s-(-2) = s+2$

$$\begin{array}{r|l}
 & s^2+4s+4 \\
s+2 & s^3+6s^2+12s+8 \\
 & -(s^3+2s^2) \\
 & 4s^2+12s+8 \\
 & -(4s^2+8s) \\
 & 4s+8 \\
 & -(4s+8) \\
 & 0
\end{array}$$

ger

$$s^3+6s^2+12s+8 = (s+2)\overbrace{(s^2+4s+4)}^{(s+2)^2} = (s+2)^3$$

så

$$F(s) = \frac{3s^2+12s+18}{(s+2)^3} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3}$$

$$= \frac{A\overbrace{(s+2)^2}^{s^2+4s+4}}{(s+2)^3} + \frac{B(s+2)}{(s+2)^3} + \frac{C}{(s+2)^3}$$

$$= \frac{As^2+(4A+B)s+(4A+2B+C)}{(s+2)^3}$$

$$
\begin{array}{l}
s^2 \\ s \\ 1
\end{array}
\begin{cases}
A & = 3 \\
4A + B & = 12 \\
4A + 2B + C & = 18
\end{cases}
\qquad
\begin{cases}
A = 3 \\
B = 0 \\
C = 6
\end{cases}
$$

så

$$F(s) = \frac{3}{s} + \frac{0}{s^2} + \frac{6}{s^3} = 3 \cdot \frac{1}{s} + 3 \cdot \frac{2!}{s^3}$$

ger

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = 3 + 3 \cdot t^2$$

Lektionsuppgift 23

$$y'' + 3y' + 2y = 2, \quad y(0) = 2, \quad y'(0) = 0$$

Laplacetransform

$$s^2 Y - s\underbrace{y(0)}_{=2} - \underbrace{y'(0)}_{=0} + 3(sY - \underbrace{y(0)}_{=2}) + 2Y = 2 \cdot \frac{1}{s}$$

$$s^2 Y + 3sY + 2Y = 2s + 6 + \frac{2}{s}$$

$$\underbrace{(s^2 + 3s + 2)}_{=(s+1)(s+2)} Y = \frac{2s^2 + 6s + 2}{s}$$

$$Y = \frac{2s^2 + 6s + 2}{s(s+1)(s+2)} \overset{\text{Hand-pålägg}}{=} \frac{1}{s} + \frac{2}{s+1} + \frac{-1}{s+2}$$

Inverstransform ger

$$\boxed{y = 1 + 2e^{-t} - e^{-2t}}$$

Lektionsuppgift 24

$$f(t) = (\theta(t) - \theta(t-1)) \cdot (t - t^2)$$

Lektionsuppgift 25

$$f(t) = (\theta(t) - \theta(t-1))(t - t^2)$$

$$= \theta(t)(t-t^2) - \theta(t-1)(t-t^2)$$

$$= \theta(t)(t-t^2) - \theta(t-1)(((t-1)+1) - ((t-1)+1)^2)$$

$$= \theta(t)(t-t^2) - \theta(t-1)((t-1)+1 - ((t-1)^2 + 2(t-1)+1))$$

$$= \theta(t)(t-t^2) - \theta(t-1)((t-1)+1 - (t-1)^2 - 2(t-1) - 1)$$

$$= \theta(t)(t-t^2) + \theta(t-1)((t-1) + (t-1)^2)$$

$$F(s) = \mathcal{L}\{f(t)\} = \frac{1}{s^2} - \frac{2}{s^3} + e^{-s}\left(\frac{1}{s^2} + \frac{2}{s^3}\right)$$

Lektionsuppgift 26

$$\begin{cases} x' - 3x + 2y = 0 \\ y' - 2x + 2y = 0 \end{cases} \qquad \begin{cases} x(0) = 1 \\ y(0) = -1 \end{cases} \qquad \begin{matrix} x = x(t) \\ y = y(t) \end{matrix}$$

Laplacetransform

$$\begin{cases} sX - \overbrace{x(0)}^{=1} - 3X + 2Y = 0 \\ sY - \underbrace{y(0)}_{=-1} - 2X + 2Y = 0 \end{cases}$$

$$\begin{cases} (s-3)X + 2Y = 1 \\ -2X + (s+2)Y = -1 \end{cases} \qquad A = \begin{bmatrix} s-3 & 2 \\ -2 & s+2 \end{bmatrix} \quad \bar{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Cramers regel

$$\det(A) = \begin{vmatrix} s-3 & 2 \\ -2 & s+2 \end{vmatrix} = (s-3)(s+2) - (-2)2$$

$$= s^2 + 2s - 3s - 6 + 4 = s^2 - s - 2 = (s+1)(s-2)$$

$$\det(A_1(\bar{b})) = \begin{vmatrix} 1 & 2 \\ -1 & s+2 \end{vmatrix} = 1\cdot(s+2) - (-1)2 = s+4$$

$$X = \frac{\det(A_1(\bar{b}))}{\det(A)} = \frac{s+4}{(s+1)(s-2)} = \frac{-1}{s+1} + \frac{2}{s-2}$$

$$\boxed{x(t) = \mathcal{L}^{-1}\{X\} = -e^{-t} + 2e^{2t}}$$

$$\det(A_2(\bar{s})) = \begin{vmatrix} s-3 & 1 \\ -2 & -1 \end{vmatrix} = (s-3)(-1) - (-2)\cdot 1 = -s+5$$

$$Y = \frac{\det(A_2(\bar{s}))}{\det(A)} = \frac{-s+5}{(s+1)(s-2)} = \frac{-2}{s+1} + \frac{1}{s-2}$$

$$\boxed{y(t) = \mathcal{L}^{-1}\{Y\} = -2e^{-t} + e^{2t}}$$

Lektionsuppgift 27

$$y = \int_0^t (t-x)\cos(2x) = t * \cos(2t)$$

$$Y = \mathcal{L}\{y\} = \mathcal{L}\{t * \cos 2t\} = \frac{1}{s^2}\cdot\frac{s}{s^2+2^2} = \frac{s}{s^2(s^2+4)}$$

Lektionsuppgift 28

$$2\cos(2t) = 2y(t) + 5\int_0^t y(t-x)\sin(2x)\,dx$$

$$2\cos(2t) = 2y(t) + 5\cdot y(t) * \sin(2t)$$

Laplacetransform

$$\cancel{2}\frac{s}{s^2+2^2} = \cancel{2}Y + 5Y\cdot\frac{\cancel{2}\,1}{s^2+2^2}$$

$$s = Y(s^2 + \underbrace{2^2}_{=4}) + 5Y$$

$$s = Y(s^2 + 5 + 4)$$

$$Y = \frac{s}{s^2+9}$$

inverstransform

$$\boxed{y(t) = \cos 3t}$$