1) $z^4+2z^3+6z^2+8z+8=0$ $\quad *$

$z=-2i$ är en rot
reella koeff $\Rightarrow$ $\bar{z}=2i$ är en rot.

även konjugatet är en rot.

$(z-(-2i))(z-2i) = (z^2+4)$ är faktor till $(*)$

polynomdivision: ger övriga faktorer

$$
\begin{array}{r|l}
 & z^2+2z+2 \\
\hline
z^2+4 & z^4+2z^3+6z^2+8z+8 \\
 & -(z^4 \qquad +4z^2) \\
\hline
 & 2z^3+2z^2+8z+8 \\
 & -(2z^3 \qquad +8z) \\
\hline
 & 2z^2 \qquad +8 \\
 & -(2z^2 \qquad +8) \\
\hline
 & 0
\end{array}
$$

$\therefore \quad z^4+2z^3+6z^2+8z+8 = (z^2+4)\cdot(z^2+2z+2)$

$z^2+2z+2=0$

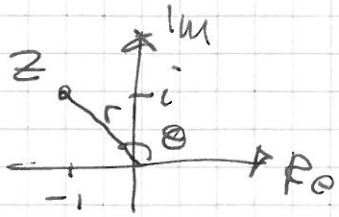
$z=-1\pm\sqrt{1-2}$

$z=-1\pm i$

Svar: $\begin{cases} z_1=-2i \\ z_2=2i \\ z_3=-1+i \\ z_4=-1-i \end{cases}$ är de 4 komplexa lösningarna

16) $w = \underbrace{(-1+i)}_{z}^{-11}$ Skriv på rektangulär form $a+ib$

$z = -1+i = r \cdot e^{i\theta}$



$r = |z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\theta = \frac{3\pi}{4}$ ur fig.

$z = \sqrt{2} \cdot e^{i\frac{3\pi}{4}} = \sqrt{2} \cdot (\cos\frac{3\pi}{4} + i \cdot \sin\frac{3\pi}{4})$

$z^{-11} = \left(\sqrt{2} \cdot e^{i\frac{3\pi}{4}}\right)^{-11} = 2^{\frac{1}{2}(-11)} \cdot e^{i\frac{3\pi(-11)}{4}}$

$= \frac{1}{2^{1/2}} \cdot e^{i(-\frac{33\pi}{4})} = \frac{1}{2^{5.5}} \cdot e^{i((\frac{32\pi}{4} + \frac{\pi}{4}))}$ (−8π)

$= \frac{1}{2^5 \cdot \sqrt{2}} \cdot e^{i(-\frac{\pi}{4})} =$

$= \frac{1}{2^5 \cdot \sqrt{2}} \cdot (\underbrace{\cos\frac{\pi}{4}}_{\frac{1}{\sqrt{2}}} + i \underbrace{\sin\frac{-\pi}{4}}_{\frac{-1}{\sqrt{2}}}) =$

$= \frac{1}{2^6} \cdot (1-i) = \frac{1}{64}(1-i)$

Alt: $w = \frac{1}{(-1+i)^{11}} = \frac{1}{32+32i} =$

$= \frac{1 \cdot (1-i)}{32\underbrace{(1+i)(1-i)}_{1^2+1^2}} = \frac{1-i}{64}$

$(-1+i)^2 = 1-2i-1 = -2i$

$(-1+i)^4 = (-2i)^2 = 4$

$(-1+i)^8 = 4 \cdot 4 = 16$

$(-1+i)^{10} = 16 \cdot (-2i) = -32i$

$(-1+i)^{11} = -32i(-1+i) = 32+32i$

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2) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = \sin 2x \qquad (*)$

(Linjär 2:a ordn. med konstanta koeff)

<u>homogenlösning:</u> $y_h = C \cdot e^{rx}$

ger karaktäristisk ekv. $r^2 - r - 2 = 0$

$r = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{2 \cdot 4}{4}}$

$r = \frac{1 \pm 3}{2} \begin{matrix} \nearrow 2 \\ \searrow -1 \end{matrix}$

$$y_h = C_1 e^{2x} + C_2 \cdot e^{-x}$$

Allmänform som HL i (*)

Ansats y_p : $A \cdot \sin 2x + B \cdot \cos 2x$

$y_p' = 2A \cdot \cos 2x - 2B \sin 2x$

$y_p'' = -4A \cdot \sin 2x - 4B \cdot \cos 2x$

} ins i (*)

$-4A \cdot \sin 2x - 4B \cdot \cos 2x - (2A \cdot \cos 2x - 2B \cdot \sin 2x) =$

$-2(A \cdot \sin 2x + B \cdot \cos 2x) = \sin 2x$

$\sin 2x$: $-4A + 2B - 2A = 1$

$\cos 2x$: $-4B - 2A - 2B = 0$

} $\begin{cases} -6A + 2B = 1 \\ -2A - 6B = 0 \end{cases}$

$\left(\begin{array}{cc|c} -6 & 2 & 1 \\ -2 & -6 & 0 \end{array}\right) \begin{matrix} \\ \textcircled{-3} \end{matrix} \sim \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 20 & 1 \end{array}\right)$ $\quad B = \frac{1}{20}$ $\quad A = -3B = \frac{-3}{20}$

$\therefore \quad y_p = \frac{-3}{20} \sin 2x + \frac{1}{20} \cdot \cos 2x$

Allmän lösning: $y(x) = y_h + y_p = C_1 \cdot e^{2x} + C_2 \cdot e^{-x} + \frac{1}{20}(-3\sin 2x + \cos 2x)$

2b) $\frac{dy}{dx}(1+x^2)+y^2=0 \qquad y(\frac{\pi}{4})=2$

ej linjär, prova separera

$$\frac{dy}{dx}(1+x^2)=-y^2$$

$$\int \frac{dy}{-y^2} = \int \frac{1}{1+x^2}\cdot dx$$

$$\int -y^{-2}\,dy = \arctan x + C$$

$$-\frac{y^{-1}}{-1} = \arctan x + C$$

$$\frac{1}{y} = \arctan x + C \qquad *$$

A.L. $$y = \frac{1}{\arctan x + C}$$

k.v. $y(\frac{\pi}{4})=2$ ins i $*$

$$\frac{1}{2} = \arctan\frac{\pi}{4} + C$$

OBS! $\tan\frac{\pi}{4}=1$

$$C = \frac{1}{2} - \underbrace{\arctan\frac{\pi}{4}}_{=?\ \approx 0{,}666}$$

$$\therefore\ y(x) = \frac{1}{\arctan x - \arctan\frac{\pi}{4} + \frac{1}{2}}$$

3a) $\sum_{k=2}^{\infty} \underbrace{\frac{1}{k \cdot \ln k}}_{a_k}$ konvergent el divergent.

$= \frac{1}{2 \cdot \ln 2} + \frac{1}{3 \ln 3} + \dots$

$a_k \longrightarrow 0$ då $k \to \infty$.

konvergens möjlig.

villkor om termernas nollgränsv. uppfyllt.

Jämforelsekriteriet:

testar konv.: nej!

$0 < a_k = \frac{1}{k \cdot \ln k} < \cancel{\begin{cases} \text{större } T \\ \text{mindre } N \end{cases}} < \frac{1}{k}$ div.

testar div.:

$\frac{1}{k \cdot \ln k} > \cancel{\begin{cases} \text{mindre } T \\ \text{större } N \end{cases}} > \frac{1}{k \cdot k}$ konv. nej!

∵ metoden kan ej användas

Integralkriteriet:

$$\int_2^{\infty} \frac{1}{x \cdot \ln x}\, dx = \left\{ \begin{array}{l} t = \ln x \\ dt = \frac{1}{x}\, dx \\ x = 2 \Rightarrow t = \ln 2 \\ x \to \infty \Rightarrow t \to \infty \end{array} \right\}$$

$$= \int_{\ln 2}^{\infty} \frac{1}{t}\, dt = \Big[\ln t \Big]_{\ln 2}^{\infty} = \text{"}\infty\text{"}$$

divergent ty gränsv. saknas.

Svar: ∵ motsvarande serie $\sum \frac{1}{k \cdot \ln k}$ är divergent.

Kvotkrit

$$\frac{a_{k+1}}{a_k} = \frac{1}{(k+1)\ln(k+1)} \cdot \frac{k \cdot \ln k}{1} = \frac{\cancel{k}}{\cancel{k}(1+\frac{1}{k})} \cdot \underbrace{\frac{\ln k}{\ln(k+1)}}_{\approx 1}$$

$\to \to 1$ kan ej avgöra konv./div.

36) $\lim_{x\to 0} \dfrac{\arctan x - x}{\ln(1+x^3)}$

Maclaurinutveckling ovh formelblad för x litet $(x \to 0)$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \mathcal{O}(x^7)$$

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} + \mathcal{O}(t^4)$$

sätt: $t = x^3$

$$\ln(1+x^3) = x^3 - \frac{(x^3)^2}{2} + \mathcal{O}(x^9)$$

im. i gränsv.

$$\lim_{x\to 0} \frac{\left[x - \frac{x^3}{3} + \frac{x^5}{5} + \mathcal{O}(x^7)\right] - x}{x^3 - \frac{x^6}{2} + \mathcal{O}(x^9)} =$$

$$= \lim_{x\to 0} \frac{x^3\left(-\frac{1}{3} + \frac{x^2}{5} + \mathcal{O}(x^4)\right)}{x^3\left(1 - \frac{x^3}{2} + \mathcal{O}(x^6)\right)} = -\frac{1}{3}$$

($x^2 \to 0$, $\mathcal{O}(x^4) \to 0$, $-\frac{x^3}{2} \to 0$, $\mathcal{O}(x^6) \to 0$)

A Svar: Gränsv. $= -\frac{1}{3}$

4. $f(t) = ?$ $\quad t \geq 0.$

$$F(s) = \frac{1}{s^2+11s+28}$$

a) faktorisera $\quad s^2+11s+28=0$

$$s = -\frac{11}{2} \pm \sqrt{\frac{121}{4} - \frac{28\cdot 4}{4}}$$

$$s = -\frac{11}{2} \pm \sqrt{\frac{9}{4}} = -\frac{11}{2} \pm \frac{3}{2} \begin{cases} -4 \\ -7 \end{cases}$$

Ansatz partialbråksuppdelning.

$$F(s) = \frac{1}{(s+4)(s+7)} \overset{!}{=} \frac{A}{s+4} + \frac{B}{s+7}$$

$$1 = A(s+7) + B(s+4)$$

$$\left.\begin{array}{ll} s: & 0 = A+B \\ s^0: & 1 = 7A+4B \end{array}\right\} \quad A=-B$$

$$1 = 7(-B)+4B = -3B$$

$$B = -\frac{1}{3} \Rightarrow A = \frac{1}{3}$$

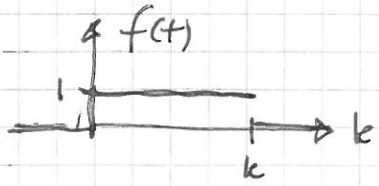
$$F(s) = \frac{\frac{1}{3}}{s+4} - \frac{\frac{1}{3}}{s+7} = \frac{1}{3}\cdot\frac{1}{s+4} - \frac{1}{3}\cdot\frac{1}{s+7}$$

$\sim e^{-4t}$ $\qquad$ $\sim e^{-7t}$

$$\underline{f(t) = \frac{1}{3}\left(e^{-4t} - e^{-7t}\right)}$$

4b) $\quad f(t) = \sigma(t) - \sigma(t-k) \qquad k>0$

$$= \begin{cases} 1 & 0<t\leq k \\ 0 & \text{f.ö.} \end{cases}$$



Laplacetransf.

$$F(s) = \frac{1}{s}\cdot e^{-0\cdot s} - \frac{1}{s}\cdot e^{-ks} = \underline{\underline{\frac{1}{s}\left(1-e^{-ks}\right)}}$$

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enligt tentamen.

5) $t \geq 0$: $y'' + 4t = e^{-t}$ $\quad y(0)=0$, $y'(0)=0$

$$y'' = e^{-t} - 4t$$

Laplacetransformering:

$$\left(s^2 \cdot Y - s \cdot \underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=0}\right) = \frac{1}{s+1} - 4 \cdot \frac{1}{s^2} = \frac{s^2 - 4(s+1)}{s^2(s+1)}$$

$$Y = \frac{s^2-4s-4}{s^2 \cdot s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s^4} + \frac{F}{s+1}$$

$$s^2 - 4s - 4 = A \cdot s^3(s+1) + B s^2(s+1) + C s(s+1) + D(s+1) + E s^4$$

s^4: $0 = A + E$ $\quad E = 1$

s^3: $0 = A + B$ $\quad A = -1$

s^2: $1 = B + C$ $\quad B = 1$

s: $-4 = C + D$ $\quad \Rightarrow C = 0$

s^0: $-4 = D$ $\quad D = -4$

$$\therefore Y = \frac{-1}{s} + \frac{1}{s^2} - \frac{4}{s^4} + \frac{1}{s+1}$$

$\frac{4}{3!} = \frac{4}{3 \cdot 2 \cdot 1} = \frac{2}{3}$

$\sim -1 \quad \sim t \quad \frac{4}{3!} \cdot \frac{3!}{s^4} \quad \sim e^{-t}$

$\sim t^3$

$$y(t) = -1 + t - \frac{2}{3}t^3 + e^{-t}$$

ej i tentamen

5) $\begin{cases} y'' + 4y = e^{-t} \\ y(0) = y'(0) = 0 \end{cases}$

$$\left(s^2Y - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=0}\right) + 4\cdot Y = \frac{1}{s+1}$$

$$Y = \frac{1}{(s^2+4)(s+1)} = \frac{As+B}{s^2+4} + \frac{C}{s+1}$$

$$1 = As(s+1) + B(s+1) + C(s^2+4)$$

$$\left.\begin{array}{ll} s^2: & 0 = A + C \\ s: & 0 = A + B \\ s^0: & 1 = B + 4C \end{array}\right\}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 \end{array}\right) \begin{matrix} \\ (-1) \\ \end{matrix} \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 4 & 1 \end{array}\right) \begin{matrix} \\ (-1) \\ \end{matrix}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 1 \end{array}\right)$$

$A = -1/5$
$B = 1/5$
$C = \frac{1}{5}$

$$Y = \frac{-\frac{1}{5}s + \frac{1}{5}}{s^2+4} + \frac{\frac{1}{5}}{s+1} = \frac{1}{5}\left[\frac{-s}{s^2+2^2} + \frac{1\cdot 2}{2s^2+2^2} + \frac{1}{s+1}\right]$$

$$y(t) = \frac{1}{5}\left[(-1)\cdot\cos 2t + \frac{1}{2}\cdot\sin 2t + e^{-t}\right]$$

6. $\frac{dy}{dx} + f(x) \cdot y = g(x) \cdot y^n$

Bernoulli-ekvation.

$$y^{-n} \cdot \frac{dy}{dx} + f(x) \cdot y^{1-n} = g(x) \quad (1)$$

Sätt: $z(x) = y^{1-n}(x) \quad \left(= \frac{y(x)}{y^n(x)} \right)$

$z'(x) = (1-n) \cdot y^{1-n-1} \cdot y'(x)$ ← inre derivatan

$= (1-n) \cdot y^{-n} \cdot y'(x)$

$\Leftrightarrow$

$\frac{z'}{1-n} = y^{-n} \cdot \frac{dy}{dx}$

ins i (1)

$\frac{1}{1-n} \cdot z' + f(x) \cdot z = g(x)$ ny linjär de! med $z(x)$

$z' + \underbrace{f(x) \cdot (1-n)}_{\text{iF...}} \cdot z = g(x)(1-n)$

Ex) $\frac{dy}{dx} + \frac{4}{x} \cdot y = x^3 \cdot y^2 \qquad y(1) = -1 \qquad x > 0.$

ej linjär. $y^{-2} \cdot \frac{dy}{dx} + y^{-1} \cdot \frac{4}{x} = x^3$ (*)

Sätt: $z(x) = y^{-1}(x) = \frac{1}{y(x)}$

$z' = -1 \cdot y^{-2} \cdot y' = -1 \cdot y^{-2} \cdot \frac{dy}{dx}$ } ins i (*)

$-z' + z \cdot \frac{4}{x} = x^3 \quad \Leftrightarrow \quad z' - \frac{4}{x} z = -x^3$

ny linjär d.e. med z

$$z'(x) - \frac{4}{x} \cdot z(x) = -x^3$$ linjär!

- if: $e^{\int -\frac{4}{x}\,dx} = e^{-4\ln x} = e^{\ln(x^{-4})} = x^{-4} = \frac{1}{x^4}$
- mult. med if

$$\underbrace{z' \cdot \frac{1}{x^4} - \frac{4}{x} \cdot \frac{1}{x^4} \cdot z}_{} = \frac{-x^3}{x^4}$$

$$\downarrow$$

$$\frac{d}{dx}\left(z \cdot \frac{1}{x^4}\right) = -\frac{1}{x}$$

- integrera

$$z(x) \cdot \frac{1}{x^4} = \int -\frac{1}{x}\,dx = -\ln|x| + C$$

$$z(x) = x^4 \cdot (C - \ln|x|)$$

Byt tillbaka!

$$z = \frac{1}{y(x)}$$

ger

$$\frac{1}{y(x)} = x^4(C - \ln|x|) \qquad (2)$$

A.L $$y(x) = \frac{1}{x^4(C - \ln|x|)}$$

b.v. $y(1) = -1$ in i (2) $\quad \frac{1}{-1} = 1\,(C - \ln 1)$

$\Leftrightarrow C = -1$

$$\therefore \quad \underline{y(x) = \frac{-1}{x^4(1 + \ln|x|)}}$$