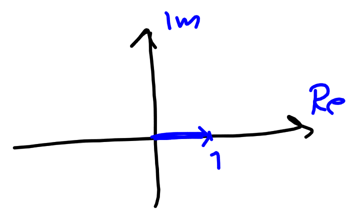
$\boxed{F4}$ Repetition komplexa tal

Rep FN 2.65a) $\boxed{z^4 = 1}$ $*$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$(x+iy)^4 = x^4 + 4x^3iy + 6x^2i^2y^2 + \dots$

Skriv på led i polär form:



Sätt: $z = r \cdot e^{i\theta} \Rightarrow$

VL: $z^4 = (r \cdot e^{i\theta})^4 = r^4 \cdot e^{i4\theta}$

HL: $1 = 1 \cdot e^{i0}$

ins i $*$

$$r^4 \cdot e^{i4\theta} = 1 \cdot e^{i0}$$

beloppen: $\begin{cases} r^4 = 1 & \Rightarrow \underline{\underline{r = 1}} \\ 4\theta = 0 + 2\pi \cdot n \end{cases}$
arg:

Glöm ej perioden

$$\begin{cases} r = 1 \\ \theta = \frac{\pi}{2} \cdot n \end{cases}$$

$\therefore \; \underline{z = r \cdot e^{i\theta} = 1 \cdot e^{i\left(\frac{\pi}{2} n\right)}}$

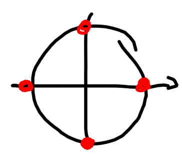
$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

$n=0$: $z_1 = 1 \cdot e^{i0} = 1 \cdot 1 = 1$

$n=1$: $z_1 = 1 \cdot e^{i\frac{\pi}{2}} = 1 \cdot (\underbrace{\cos\frac{\pi}{2}}_{0} + i \underbrace{\sin\frac{\pi}{2}}_{1}) = i$

$n=2$: $z_2 = -1$

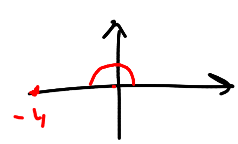
$n=3$: $z_3 = -i$



Svar: $z_{1,2} = \pm 1 \qquad z_{3,4} = \pm i$

b) $z^4 = \underbrace{-4}$

$r^4 \cdot e^{i4\theta} =$ polär form ger: $4 \cdot e^{i\pi}$



Ö 2.35) $|e^{x+iy}|$ där $x, y \in \mathbb{R}$.

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = \underline{\underline{1}}$$

$$e^{x+iy} = e^x \cdot e^{iy}$$

$$|e^x \cdot e^{iy}| = |e^x| \cdot \underbrace{|e^{iy}|}_{1} = \underline{\underline{e^x}} \cdot 1$$

$$|e^{iy}| = |\cos y + i\sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1$$

- - - - - - - - -

Tentamensuppg:

Bestäm $z \in \mathbb{C}$ där $e^z - 1 + i = 0$ (2p)

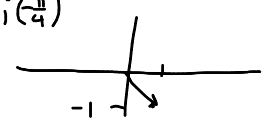
$$e^z = 1 - i$$

Sätt: $\boxed{z = x + iy}$

Skriv HL i polär form:

$$1 - i = \sqrt{2} \cdot e^{i\left(-\frac{\pi}{4}\right)}$$

ur fig.



$$e^z = 1 - i$$

$$e^{x+iy} = \sqrt{2} \cdot e^{i\left(-\frac{\pi}{4}\right)}$$

$$e^x \cdot e^{iy} = \sqrt{2} \cdot e^{i\left(-\frac{\pi}{4}\right)}$$

belopp: $e^x = \sqrt{2}$ $\Rightarrow$ $x = \underline{\ln\sqrt{2}} = \frac{1}{2}\ln 2$

arg: $y = -\frac{\pi}{4} + 2\pi n$

$$\therefore\ z = x + iy = \underline{\ln\sqrt{2} + i\left(-\frac{\pi}{4} + 2\pi \cdot n\right)}$$

$$\boxed{e^{-\pi i} + 1 = 0}$$

"Världens vackraste ekvation"

Ö 2.27) $\cos 3\theta$ uttryck med $\cos\theta$ och $\sin\theta$.

de Moivres formel: $z^n = (r\cdot e^{i\theta})^n = r^n\cdot e^{in\theta} =$

$= r^n\cdot(\cos n\theta + i\sin n\theta)$

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$$

$z = \underbrace{\cos 3\theta}_{Re(z)} + i\,\underbrace{\sin 3\theta}_{Im(z)} = \{\text{de Moivres}\} =$

$= (\cos\theta + i\sin\theta)^3 = (\quad)\cdot(\quad)\cdot(\quad) =$

$= 1\cdot\cos^3\theta + 3\cdot\cos^2\theta\cdot i\sin\theta + 3\cos\theta\,\underbrace{i^2}_{-1}\sin^2\theta + \underbrace{i^3}_{-i}\sin^3\theta =$

$= \cos^3\theta - 3\cos\theta\cdot\sin^2\theta + i\cdot[3\cos^2\theta\cdot\sin\theta - \sin^3\theta]$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$\therefore \cos 3\theta = \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$

$= 4\cos^3\theta - 3\cos\theta$.

ex) $\sqrt{i} = ?$ $\qquad z = x + iy = r \cdot e^{i\theta}$

$\sqrt{i} = z$ $\qquad$ kvadrera

$i = z^2$

<u>Sätt</u>: $z = r \cdot e^{i\theta}$ $\qquad$ polär form $\Rightarrow z^2 = r^2 \cdot e^{i2\theta}$

$i = 1 \cdot e^{i\frac{\pi}{2}}$ $\qquad$ – " –

$\therefore \quad r^2 \cdot e^{i2\theta} = 1 \cdot e^{i\frac{\pi}{2}}$

Beloppen: $\begin{cases} r^2 = 1 & \Rightarrow \quad r = 1 \\ 2\theta = \frac{\pi}{2} + 2\pi \cdot n & \Rightarrow \quad \theta = \frac{\pi}{4} + \pi \cdot n \end{cases}$
arg:

$\therefore \quad z = 1 \cdot e^{i\left(\frac{\pi}{4} + \pi n\right)}$

$n=0$: $\quad z_1 = e^{i\frac{\pi}{4}} = \underbrace{\cos\frac{\pi}{4}}_{\frac{1}{\sqrt{2}}} + i\underbrace{\sin\frac{\pi}{4}}_{\frac{1}{\sqrt{2}}} = \underline{\underline{\frac{1}{\sqrt{2}}(1+i)}}$

$n=1$: $\quad z_1 = e^{i\frac{5\pi}{4}} = \underbrace{\cos\frac{5\pi}{4}}_{-\frac{1}{\sqrt{2}}} + i\underbrace{\sin\frac{5\pi}{4}}_{-\frac{1}{\sqrt{2}}} = \underline{\underline{-\frac{1}{\sqrt{2}}(1+i)}}$

Svar: $\underline{\underline{\sqrt{i} = \pm\frac{1}{\sqrt{2}}(1+i)}}$

<u>Alt</u>: $\quad z = x + iy \Rightarrow z^2 = x^2 + 2xyi - y^2 \equiv i$

$$\frac{\sqrt{3}-i}{(1+i)^3} =$$

Tälj. $\sqrt{3}-i = r\cdot e^{i\theta}$

$r = |\sqrt{3}-i| = \sqrt{\sqrt{3}^2+(-1)^2} = 2$

$\theta = \arctan \frac{Im}{Re} = \underbrace{\arctan\left(\frac{-1}{\sqrt{3}}\right)}_{-\frac{\pi}{6}}$

Nämn. $1+i = r_1\cdot e^{i\theta_1}$

$r_1 = |1+i| = \sqrt{2}$

$\theta_1 = \arctan 1 = \frac{\pi}{4}$

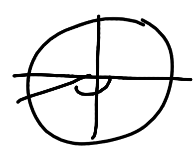
$$= \frac{2\cdot e^{i\left(-\frac{\pi}{6}\right)}}{\left[\sqrt{2}\cdot e^{i\frac{\pi}{4}}\right]^3} =$$

$$= \frac{2\cdot e^{i\left(-\frac{\pi}{6}\right)}}{2^{\frac{3}{2}}\cdot e^{i\left(\frac{3\pi}{4}\right)}} =$$

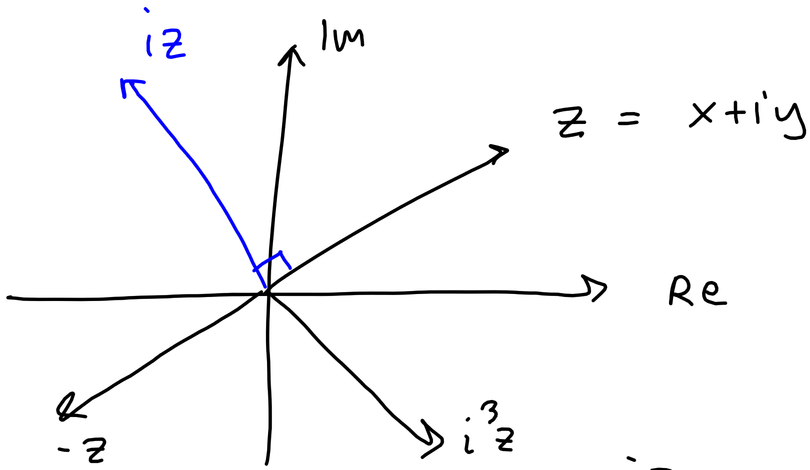
$$= 2^1\cdot 2^{-3/2}\cdot e^{i\left(-\frac{\pi}{6}-\frac{3\pi}{4}\right)} =$$

$$= 2^{-1/2}\cdot e^{i\left(\frac{-4\pi-18\pi}{24}\right)} = 2^{-1/2}\cdot e^{i\left(-\frac{\cancel{22}\,11\,\pi}{\cancel{24}\,12}\right)} =$$

$$= \frac{1}{\sqrt{2}}\cdot\left(\underbrace{\cos\left(-\frac{11\pi}{12}\right)} + i\,\underbrace{\sin\left(-\frac{11\pi}{12}\right)}\right) \approx$$



$$\approx \underline{\underline{-0{,}68 - i\,0{,}18}}$$



- $i^2 \cdot z = -z$
- $i^3 \cdot z = i^2 \cdot iz = -iz$
- $i^4 \cdot z = i^2 \cdot i^2 \cdot z = z$

- iz

mult. med i medför vridning $\frac{\pi}{2}$ moturs i komplexa talplanet.