$\boxed{\text{F15-16}}$ Potensserier.

FN 10.1 c) $$\sum_{k=1}^{\infty} \underbrace{\frac{1}{k^2+2k}}_{a_k} = \frac{1}{1+2} + \frac{1}{4+4} + \frac{1}{9+6} + \cdots$$

$$= \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \cdots$$

$\sum_{k=1}^{\infty} \frac{1}{k^2}$ konvergent ty p-serie med $p = 2 > 1$

① Integral kriteriet:

$$\int_1^{\infty} \frac{1}{x^2+2x}\,dx \text{ konv} \Rightarrow \sum a_k \text{ konv}$$

② Jämförelsekrit.

$$0 < a_k < \frac{1}{k^2+2k} < \frac{1}{k^2+0} = \frac{1}{k^2} = b_k$$

$\sum \frac{1}{k^2}$ konv

③ Kvotkriteriet:

$$\frac{a_{k+1}}{a_k} = \frac{\frac{1}{(k+1)^2+2(k+1)}}{\frac{1}{k^2+2k}} = \frac{\frac{1}{(k+1)\cdot[k+1+2]}}{\frac{1}{k(k+2)}} = \frac{k(k+2)}{(k+1)(k+3)} =$$

$$= \frac{\cancel{k^2}\cdot\left(1+\frac{2}{k}\right)}{\cancel{k^2}\cdot\left(1+\frac{1}{k}\right)\left(1+\frac{3}{k}\right)} \to \frac{1+0}{(1+0)(1+0)} = 1 \quad \text{då } k \to \infty$$

∴ kan ej avgöra konvergens med kvotkriteriet.

forts summan

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2+2k}$$

Partialbråksuppdelning

Ansats.

$$\frac{1}{k^2+2k} = \frac{1}{k(k+2)} = \frac{A}{k} + \frac{B}{k+2}$$

$$1 = A(k+2) + Bk$$

$$\left.\begin{aligned} k:&\quad 0 = A+B \\ k^0:&\quad 1 = 2A \end{aligned}\right\} \quad \begin{aligned} B &= -1/2 \\ A &= 1/2 \end{aligned}$$

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$$S_n = \sum_{k=1}^{n}\left(\frac{1/2}{k} - \frac{1/2}{k+2}\right) = \frac{1}{2}\cdot\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+2}\right) =$$

$$= \frac{1}{2}\left[\left(\frac{1}{1} - \cancel{\frac{1}{1+2}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{6}\right) + \cancel{} \dots + \left(\frac{1}{n} - \frac{1}{n+2}\right)\right]$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} + 0 - 0 \dots\dots - \frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$\lim_{n\to\infty} S_n = S = \frac{1}{2}\left(1 + \frac{1}{2} - 0 - 0\right) = \underline{\underline{\frac{3}{4}}}$$

FN

10.2

a) $\sum_{k=1}^{\infty} \frac{2}{3^k} = \frac{2}{3} + \frac{2}{3^2} + \dots$

$= 2 \cdot \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k =$

$= 2 \cdot \left(\frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \dots \right)$

$= 2 \cdot \frac{1}{3} \left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \dots \right)$

$\cdot\frac{1}{3}$ $\cdot\frac{1}{3}$ $\cdot\frac{1}{3}$

konstant kvot $= \frac{1}{3} < 1$

$\Rightarrow$ Geometrisk summa.

sats 10.3 sid 439

$= \frac{2}{3} \cdot \underbrace{\qquad\qquad}_{\frac{1}{1 - 1/3}} =$

$= \frac{2}{3} \cdot \frac{1}{2/3} = \underline{\underline{1}}$

$\therefore \sum_{k=1}^{\infty} \frac{2}{3^k}$ konvergerar mot summan 1.

Konvergenskriterier:

$$\sum_{n=1}^{\infty} a_n$$

1) integralkrit.
2) Jämförelsekrit.
3) Kvotkriteriet
4) Rotkriteriet

Rotkriteriet:

Om $(a_n)^{\frac{1}{n}} = \sqrt[n]{a_n} \longrightarrow L < 1$ då $n \to \infty$

är serien konvergent.

dvs: $\boxed{0 \leq \lim_{n\to\infty} \sqrt[n]{a_n} < 1}$ är serien konv.

" " > 1 är " div.

" $= 1$ kan konv. /div inte avgöras.

ex) $\sum_{n=1}^{\infty} \underbrace{\frac{1}{n^n}}_{= a_n}$

Rotkriteriet:

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{1}{n^n}} = \left(\frac{1}{n^n}\right)^{\frac{1}{n}} = \frac{1^{\frac{1}{n}}}{(n^n)^{\frac{1}{n}}} = \frac{1}{n^{\frac{n}{n}}} = \frac{1}{n}$$

$$\longrightarrow 0 < 1 \text{ då } n \to \infty$$

$\therefore \sum a_n$ är konvergent enl. rotkriteriet.

ex) $\sum_{n=1}^{\infty} \underbrace{\frac{2^{n+1}}{n^n}}_{=a_n}$. konv/div ?

Rotkrit.

$$\sqrt[n]{a_n} = a_n^{\frac{1}{n}} = \left(\frac{2^{n+1}}{n^n}\right)^{\frac{1}{n}} = \frac{(2^n \cdot 2)^{\frac{1}{n}}}{(n^n)^{\frac{1}{n}}} =$$

$$\boxed{(a^b)^c = a^{b\cdot c}} \quad \boxed{(a\cdot b)^c = a^c \cdot b^c}$$

$$= \frac{(2^n)^{\frac{1}{n}} \cdot 2^{\frac{1}{n}}}{n} = \frac{2 \cdot 2^{\frac{1}{n}}}{n} \to 0 < 1$$

$2^{\frac{1}{n}} \to 2^0 = 1$

då $n \to \infty$

$\therefore$ $\sum a_n$ är konv. enl. rot.kriteriet.

FN 10.3 a)

Partialbråksuppdelning → Teleskopserie.
för att bestämma summan

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)}$$

ansats.

$$\frac{1}{k(k+1)(k+2)} = \frac{1/2}{k} + \frac{-1}{k+1} + \frac{1/2}{k+2}$$

(Handpåläggning)
ger $A = 1/2 = C$
$B = -1$

$$S_n = \sum_{k=1}^{n}\left(\frac{1/2}{k} - \frac{1}{k+1} + \frac{1/2}{k+2}\right) = \frac{1}{2}\cdot\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2}\right) =$$

$$= \frac{1}{2}\left[\left(\frac{1}{1} - \frac{2}{2} + \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{2}{6} + \frac{1}{7}\right) + \dots + \left(\frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right)\right].$$

0 0 0

$$= \frac{1}{2}\left(1 - 1 + \frac{1}{2} + 0 + \frac{1}{n+1} - \frac{2}{n+1} + \frac{1}{n+2}\right).$$

då $n \to \infty$:

$$\lim_{n\to\infty} S_n = \frac{1}{2}\left(1 - 1 + \frac{1}{2} + 0\right) = \underline{\underline{\frac{1}{4}}}$$

Ex) $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{(2n)!}{(n!)^2}}_{a_n}$

Kvotkriteriet – bra vid fakulteter.

$$\frac{a_{n+1}}{a_n} = \frac{\frac{[2(n+1)]!}{[(n+1)!]^2}}{\frac{(2n)!}{(n!)^2}} = \frac{(2n+2)! \cdot n!\, n!}{(n+1)!\,(n+1)! \cdot (2n)!} =$$

$$(2(n+1))! = (2n+2)!$$
$$= (2n+2)\cdot(2n+1)\cdot \underbrace{(2n) \cdot \ldots \cdot}_{(2n)!} = (2n+2)\cdot(2n+1)(2n)!$$

$$(n+1)! = (n+1)\cdot \underbrace{n\cdot(n-1)\cdot \ldots \cdot 1}_{=n!}$$
$$= (n+1)\cdot n!$$

$$4! = 4\cdot 3\cdot 2\cdot 1 = 24$$
$$= 4\cdot 3!$$

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)(2n+1)\cancel{(2n)!}\cdot \cancel{n!}\cdot\cancel{n!}}{(n+1)\cdot\cancel{n!}\cdot(n+1)\cancel{n!}\quad \cancel{(2n)!}} =$$

$$= \frac{2\cancel{(n+1)}(2n+1)}{\cancel{(n+1)}(n+1)} = \frac{2\cancel{n}(2+\frac{1}{n})}{\cancel{n}\cdot(1+\frac{1}{n})} \to 2\cdot\frac{(2+0)}{1+0} = 4$$

$$\not< 1$$

∴ serien är divergent enl kvot.krit.

ty $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 4 > 1.$

Potensserier

t.ex. $\begin{cases} \text{Maclaurin serie} \\ \text{Taylor serie} \end{cases}$

En serie med en variabel x i termerna som bildar ett "oändligt polynom".

$$\boxed{\sum_{k=0}^{\infty} c_k \cdot x^k = c_0 + c_1 \cdot x + c_2 \cdot x^2 + c_3 x^3 + \ldots}$$

Geometrisk serie är en potensserie med $c_k = 1$ $\forall k$.

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots$$

$\cdot x \quad \cdot x \quad \cdot x$

En avhuggen (trunkerad) potensserie bildar ett polynom

- - -

När konvergerar potensserien, dvs för vilka x?

- Hitta konvergensområde till x.
- Bestämma konvergensradie (R)

Till varje potensserie $\sum_{k=0}^{\infty} c_k \cdot x^k$ finns ett tal R (konvergensradien) så att

serien är konvergent $\forall \ |x| < R$

" divergent $\forall \ |x| > R$

Om $R = \infty \Rightarrow$ serien konv för alla x

$R = 0 \Rightarrow$ " omm $x = 0$

Då $x = \pm R$ kan ej konv/div avgöras.

ex) $$\sum_{k=0}^{\infty} \underbrace{5^k \cdot x^k}_{a_k} = 1 + 5x + 25x^2 + \cdots$$

<u>Kvotkrit.</u>

$$\frac{a_{k+1}}{a_k} = \frac{5^{k+1} \cdot x^{k+1}}{5^k \cdot x^k} = \frac{5^k \cdot 5 \cdot x^k \cdot x}{5^k \cdot x^k} = 5x$$

konvergerar om $|5x| < 1$ enl. kvotkrit.

$|x| < \frac{1}{5} = R$

- konvergensradie: $R = \frac{1}{5}$
- konvergensområde: $-\frac{1}{5} < x < \frac{1}{5}$