$\boxed{F17}$ Potensserier forts.
Taylor- och MacLaurinserier.

FN 10.21 g) $\underbrace{\sum_{k=0}^{\infty} \frac{k!}{(2k)!}}_{a_k} x^k$ potensserie då serien innehåller en variabel x.

$$= \frac{0!}{0!}x^0 + \frac{1}{2\cdot 1}x + \frac{2\cdot 1}{4\cdot 3\cdot 2\cdot 1}x^2 + \dots$$

$$= 1 + \frac{1}{2}x + \frac{1}{12}x^2 + \dots$$

För vilka x är serien konvergent?

Kvotkriteriet

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{\frac{(k+1)!}{[2(k+1)]!}x^{k+1}}{\frac{k!}{(2k)!}x^k}\right| = \left|\frac{(k+1)\cdot \cancel{k!}}{(2k+2)!}\cancel{x^k}\cdot x \frac{(2k)!}{\cancel{k!}\cdot \cancel{x^k}}\right| =$$

$$= \left|\frac{(k+1)\cancel{(2k)!}\cdot x}{(2k+2)\cdot(2k+1)\cancel{(2k)!}}\right| = \frac{\cancel{(k+1)}\ |x|}{2\cancel{(k+1)}(2k+1)} \to$$

$$(2k+2)! = (2k+2)(2k+1)\cdot \underbrace{(2k)\cdot(2k-1)\cdot\ldots\cdot 3\cdot 2\cdot 1}_{=(2k)!}$$

$$\to 0\cdot|x| < 1$$ Krav på konvergens enl kvot krit.

$$|x| < \infty$$

då $k \to \infty$.

$$L = \lim \left|\frac{a_{k+1}}{a_k}\right|$$ där $\sum_{k=0}^{\infty} a_k \cdot x^k$

Kvotkriteriet konvergerar då

$$|L\cdot x| < 1$$

$$|x| < \frac{1}{L} = R$$ = konvergensradien.

Ö 10.10 d) $\displaystyle\sum_{k=1}^{\infty} \underbrace{\frac{\ln k}{k} \cdot x^k}_{a_k}$ För villka x är serien konvergent?

Kvotkriteriet:

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{\frac{\ln(k+1)}{k+1}}{\frac{\ln k}{k}} |x| = \frac{\ln(k+1)}{\ln k} \cdot \frac{k}{(k+1)} \cdot |x| =$$

$$= \frac{\ln k(1+\frac{1}{k}) \cdot \cancel{k}}{\ln k \ \cancel{k}(1+\frac{1}{k})} |x| = \frac{\ln k + \ln(1+\frac{1}{k})}{\ln k \cdot (1+\frac{1}{k})} \cdot |x|$$

$$= \frac{\cancel{\ln k} \cdot |x|}{\cancel{\ln k} \cdot (1+\frac{1}{k})} + \frac{\ln(1+\frac{1}{k})}{\ln k \cdot (1+\frac{1}{k})} \cdot |x| =$$

$$\frac{|x|}{1+\frac{1}{k}} + \underbrace{\frac{\overbrace{\ln(1-\frac{1}{k})}^{\ln 1 = 0}}{1+\frac{1}{k}} \cdot |x|}_{\to 0} \rightarrow \frac{|x|}{1} < 1$$

$$\underline{\underline{|x| < 1}}$$

då $k \to \infty$.

FN 10.21 b) $\sum_{k=0}^{\infty} k!\, x^k = 1 + 1\cdot x + 2\cdot x^2 + 3!\, x^3 + \dots$

Kvotkriteriet

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{(k+1)!}{k!}|x| = \frac{(k+1)\cancel{k!}}{\cancel{k!}}|x| = (k+1)|x|$$

$\longrightarrow \infty \cdot |x| < 1$ enl kvotkrit.

då $k \to \infty$

$\boxed{|x| < 0}$ $R = 0$

Serien bara konvergent för $x = 0$.

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ex) $\sum_{n=1}^{\infty} \underbrace{\frac{1}{n\cdot 2^n}(x-5)^n}_{a_n} = \frac{1}{1\cdot 2}(x-5) + \frac{1}{2\cdot 2^2}(x-5)^2 + \dots$

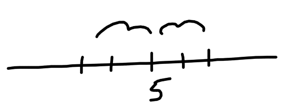
Kvotkrit

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{(n+1)\cdot 2^{n+1}}(x-5)^{n+1}}{\frac{1}{n\cdot 2^n}(x-5)^n}\right| = \frac{n\cdot \cancel{2^n}\cdot}{(n+1)\cdot\cancel{2^n}\cdot 2}|x-5| =$$

$$= \frac{\cancel{n}}{\cancel{n}(1+\frac{1}{n})\cdot 2}|x-5| \longrightarrow \frac{1}{2}\cdot|x-5| < 1$$

då $n \to \infty$

$|x-5| < 2$



$3 < x < 7$

Konvergensradie $\underline{\underline{R=2}}$.

ty: $|x-5| < 2$

$-2 < x-5 < 2$

$-2+5 < x < 2+5$

$\underline{3 < x < 7}$

$$3\cdot\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k = 3\cdot\underbrace{\left(1+\frac{2}{3}+\left(\frac{2}{3}\right)^k+\cdots\right)}$$

$$3\cdot\frac{1}{1-\frac{2}{3}} = 3\cdot\frac{1}{1/3} = 9.$$

Geometrisk summa med kvoten $2/3$.

Potensserie:

$$f(x) = \sum_{k=0}^{\infty} c_k \cdot x^k = c_0 + c_1 x + c_2 \cdot x^2 + \dots$$

konvergerar för $|x| < R$

$$* \; f'(x) = \sum_{k=1}^{\infty} k \cdot c_k \cdot x^{k-1} = c_1 + 2 \cdot c_2 \cdot x + 3 \cdot c_3 \cdot x^2 + \dots$$

$$F(x) = \sum_{k=0}^{\infty} \frac{c_k \cdot x^{k+1}}{k+1} = c_0 \cdot x + \frac{c_1 x^2}{2} + \frac{c_2 \cdot x^3}{3} + \dots$$

Byt index i summan

sätt: $i = k-1$ $\quad \sum_{\underbrace{k-1}_{i}=0}^{\infty} (i+1) \cdot c_{i+1} \, x^i = 1 \cdot c_1 x^0 + 2 c_2 x + \dots$

sätt: $k = i$ $\quad f'(x) = \sum_{k=0}^{\infty} (k+1) c_{k+1} \cdot x^k = 1 \cdot c_1 \cdot x^0 + 2 c_2 x + 3 c_3 x^2 + \dots$

$$f'(x) = \sum_{k=1}^{\infty} k \; c_k \cdot x^{k-1}$$

identiska serier

bara olika index.

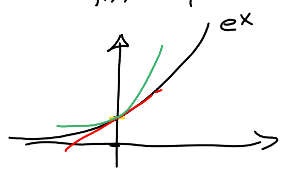
Taylor och MacLaurin serier

Deriverbara funktioner (e^x, $\sin x$, $\sqrt{1+x}$, $\ln x$, $\arctan x$) kan uttryckas med Taylor- eller MacLaurin serier.

ex) $f(x) = e^x$

Approximera med polynom av grad 3 för x nära 0.

$$f(x) \approx p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 = \sum_{k=0}^{3} c_k \cdot x^k$$



Vill att polynomet har samma

a) funktionsv. då $x=0$: $f(0)=p(0)$

b) lutning " : $f'(0)=p'(0)$

c) krökning " : $f''(0)=p''(0)$

d) 3:e deriv " $f'''(0)=p'''(0)$.

funktion	polynom
$f(x)=e^x$	$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$

a) $f(0)=1$ $\quad p(0)=c_0$

$\therefore \underline{1=c_0}$ $\quad \boxed{c_0 = f(0)}$

b) $f'(x)=e^x$ $\quad p'(x) = c_1 + 2c_2 x + 3c_3 x^2$

$f'(0)=1$ $\quad p'(0)=c_1$

$\therefore \underline{1=c_1}$ $\quad \boxed{c_1 = f'(0)}$

c) $f''(x)=e^x$ $\quad p''(x) = 2c_2 + \overset{3\cdot 2}{6} c_3 x$

$f''(0)=1$ $\quad p''(0)=2c_2$

$\therefore 1 = 2c_2 \iff \underline{c_2 = \frac{1}{2}}$ $\quad \boxed{c_2 = \frac{f''(0)}{2}}$

$f''(0) = 2\cdot c_2$

d) $f'''(x)=e^x$ $\quad p'''(x) = 6c_3$

$\underline{f'''(0)=1}$ $\quad \underline{p'''(0) = 6c_3}$

$\therefore 1 = 6c_3 \iff c_3 = \frac{1}{6}$ $\quad \boxed{c_3 = \frac{f'''(0)}{3\cdot 2}}$

$f''(0) = 6\cdot c_3$

$$\therefore p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$
$$= \underline{1 + 1\cdot x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3}$$

test. $\underline{x=1}$

$f(1) = e^1 \approx 2{,}718281 82\ldots$

$p(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} \approx 2{,}67$

$\underline{x=0{,}1}$

$f(0{,}1) = e^{0{,}1} = 1{,}105171\ldots$

$p(0{,}1) \approx 1{,}105167\ldots$

Maclaurinserie – för x värden nära 0.

$$\boxed{f(x) = f(0) + f'(0)\cdot x + \frac{f''(0)}{2}\cdot x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}\cdot x^4 + \dots}$$