$\boxed{F18}$ · MacLaurin- och Taylorserier
· Serielösning av ordinära diff. ekv (ODE).

forts. F17

$e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$ 3:e grads polynom Maclaurin polynom.

$e^x = \underbrace{1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots}_{\text{Maclaurin serie för } x \text{ nära } 0.}$

$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \mathcal{O}(x^4)$

stora ordo "big-Oh"

$\mathcal{O}(x^4)$ är av storleksordning högst x^4

$\mathcal{O}(x^4) \to 0$ då $x \to 0$.

$f(x) = f(0) + f'(0)\cdot x + \frac{f''(0)}{2!}\cdot x^2 + \frac{f'''(0)}{3!}\cdot x^3 + \frac{f^{(4)}(0)}{4!}\cdot x^4 + \mathcal{O}(x^5)$

tal.

Ger en Maclaurin serie

ex) $f(x) = \sin x$ Maclaurin serie för x nära 0.

$f(x) = \sin x$
$f'(x) = \cos x$
$f''(x) = -\sin x$
$f'''(x) = -\cos x$
$f^{(4)}(x) = \sin x$

$f(0) = \sin 0 = 0$
$f'(0) = \cos 0 = 1$
$f''(0) = -\sin 0 = 0$
$f'''(0) = -\cos 0 = -1$
$f^{(4)}(0) = 0$

$$\therefore \boxed{f(x) = f(0) + f'(0)\cdot x + \frac{f''(0)}{2!}\cdot x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \mathcal{O}(x^5)}$$

$\sin x = 0 + 1\cdot x + 0 - \frac{1}{3!}\cdot x^3 + 0 + \mathcal{O}(x^5)$.

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$

Se tabellsamling!

Ex) $\lim_{t\to 0} \dfrac{\sin t - t}{t^3}$

Maclaurin utv. av $\sin t = t - \dfrac{t^3}{3!} + \dfrac{t^5}{5!} + O(t^7)$

(enl. tabell)

$$\lim_{t\to 0} \frac{\cancel{t} - \frac{t^3}{3\cdot 2\cdot 1} + \frac{t^5}{5\cdot 4\cdot 3\cdot 2\cdot 1} + O(t^7) - \cancel{t}}{t^3} =$$

→ 0 då $t\to 0$

$$= \lim_{t\to 0} \frac{\cancel{t^3}\left(-\frac{1}{6} + \underbrace{\frac{t^2}{120}}_{\to 0} + O(t^4)\right)}{\cancel{t^3}} = -\frac{1}{6}$$

Tentauppg.

Ex) $\lim_{x\to 0} \dfrac{\arctan x - x}{\ln(1+x^3)}$

Maclaurinserier ty x litet ($\to 0$).

- $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} + O(x^7)$
- $\ln(1+t) = t - \dfrac{t^2}{2} + \dfrac{t^3}{3} - \dfrac{t^4}{4} + O(t^5)$

sätt $t = x^3$ ger

$$\ln(1+x^3) = x^3 - \frac{(x^3)^2}{2} + \frac{(x^3)^3}{3} + O(x^{12})$$

$$\therefore \lim_{x\to 0} \frac{\cancel{x} - \frac{x^3}{3} + \frac{x^5}{5} + O(x^7) - \cancel{x}}{x^3 - \frac{x^6}{2} + O(x^9)} =$$

$$= \lim_{x\to 0} \frac{\cancel{x^3}\left(-\frac{1}{3} + \underbrace{\frac{x^2}{5}}_{\to 0} + \underbrace{O(x^4)}_{\to 0}\right)}{\cancel{x^3}\left(1 - \underbrace{\frac{x^3}{2}}_{\to 0} + \underbrace{O(x^6)}_{\to 0}\right)} = -\frac{1}{3} \qquad (2p).$$

Ex) $\lim_{x \to 0} \frac{\arctan x - x}{\sin x - x}$

$$= \lim_{x \to 0} \frac{\left(\cancel{x} - \frac{x^3}{3} + \frac{x^5}{5} + O(x^7)\right) - \cancel{x}}{\left(\cancel{x} - \frac{x^3}{3!} + \frac{x^5}{5!} + O(x^7)\right) - \cancel{x}} =$$

$$= \lim_{x \to 0} \frac{\cancel{x^3} \cdot \left(-\frac{1}{3} + \frac{x^2}{5} + O(x^4)\right)}{\cancel{x^3} \cdot \left(-\frac{1}{3 \cdot 2 \cdot 1} + \frac{x^2}{5!} + O(x^4)\right)} = \frac{-\frac{1}{3}}{-\frac{1}{6}} = \underline{\underline{2}}$$

Något om ordo för x nära 0.

$O(x^4)$ – funktion av storleksordn. högst x^4

$O(x^n) \to 0$ då $x \to 0$ $\quad n \in \mathbb{Z}_+$

ingår i

- $O(x^2) + O(x^3) = O(x^2)$
- $5x^3 \cdot O(x) = O(x^4)$
- $\frac{O(x^3)}{x} = O(x^2)$
- $O(x^3) - O(x^3) = O(x^3)$
- $O(x^2) + O(x^2) = O(x^2)$
- $O(x^2) \cdot O(x^3) = O(x^{2+3}) = O(x^5)$

Taylorserie: Taylorpolynom för x nära a

$$\boxed{f(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}\cdot(x-a)^2 + \frac{f'''(a)}{3!}\cdot(x-a)^3 + \mathcal{O}((x-a)^4)}$$

ex) $f(x) = \arctan x$. Taylorpolynom av grad 5 kring $x=1$

$$\arctan x \approx f(1) + f'(1)\cdot(x-1) + \frac{f''(1)}{2}(x-1)^2 + \dots + \frac{f^{(5)}(1)}{5!}\cdot(x-1)^5$$

$f(x) = \arctan x$

$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$

$f''(x) = -(1+x^2)^{-2}\cdot 2x$

$= \frac{-2x}{(1+x^2)^2}$

$\vdots$

$f(1) = \arctan 1 = \frac{\pi}{4}$

$f'(1) = \frac{1}{1+1} = \frac{1}{2}$

$f''(1) = \frac{-2}{(1+1)^2} = -\frac{2}{4} = -\frac{1}{2}$

$f'''(1) = 1/2$

$f^{(4)}(1) = 0$

$f^{(5)}(1) = -3$

$$\therefore \arctan x \approx f(1) + \frac{f'(1)}{1!}(x-1)^1 + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}\cdot(x-1)^4 + \dots$$

$$\approx \frac{\pi}{4} + \frac{1}{2}(x-1) + \left(-\frac{1}{2}\right)\cdot\frac{1}{2!}(x-1)^2 + \frac{1}{2}\cdot\frac{1}{3!}(x-1)^3 + 0 - \frac{3}{5!}(x-1)^5$$

$$\arctan x \approx \underbrace{\frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3 - \frac{1}{40}(x-1)^5}_{\text{Taylorpolynom av grad 5 kring } x=1}$$

Om $x = 0{,}9$

$$\arctan 0{,}9 \approx \frac{\pi}{4} + \frac{1}{2}(0{,}9-1) - \frac{1}{4}$$

$$\approx \frac{\pi}{4} + \frac{1}{2}(-0{,}1) - \frac{1}{4}(0{,}01) + \frac{1}{12}(-0{,}001) - \frac{1}{40}(-0{,}00001)$$

$$\approx 0{,}73281508$$

miniräknare: $0{,}73281510$

$f(x) = e^x \cdot \cos x$ (Maclaurinuðu.)

$f(0) = 1$

$f'(x) = e^x \cdot \cos x + e^x \cdot (-\sin x)$

$f'(0) = 1$

$f''(x) = \cancel{e^x \cdot \cos x} + e^x(-\sin x) + e^x(-\sin x) + \cancel{e^x(-\cos x)}$

$f''(0) = e^0(-2\sin 0) = 0$

ins i $\quad f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!} \cdot x^2 + \dots$

Alt:

$$e^x = 1 + x + \frac{x^2}{2!} + O(x^3)$$

$$\cos x = 1 - \frac{x^2}{2} + O(x^4)$$

$$f(x) = e^x \cdot \cos x = \left(1 + x + \frac{x^2}{2} + O(x^3)\right) \cdot \left(1 - \frac{x^2}{2} + O(x^4)\right)$$

$$= 1 - \cancel{\frac{x^2}{2}} + O(x^4) + x - \frac{x^3}{2} + O(x^5) + \cancel{\frac{x^2}{2}} - \frac{x^4}{4} + O(x^6) + O(x^3)$$

$$= 1 + x + O(x^3)$$

"svelgjast af"

Potensserielösningar. Se "Material till L18" i Fronter

Lösa o.d.e. med potensserier.

ex) $y''(x) + y = 0$ $\quad y(0) = 1 \Rightarrow c_0 = 1$; $y'(0) = 0 \Rightarrow c_1 = 0$

Sätt: $y = c_0 + c_1 \cdot x + c_2 x^2 + \dots = \sum_{k=0}^{\infty} c_k \cdot x^k$

$y' = c_1 + 2c_2 x + 3c_3 x^2 = \sum_{k=1}^{\infty} k\, c_k \cdot x^{k-1}$

$y'' = 2c_2 + 3 \cdot 2 \cdot c_3 x = \sum_{k=2}^{\infty} k \cdot (k-1) \cdot c_k \cdot x^{k-2}$

ins i d.e.

$$\sum_{k=2}^{\infty} k(k-1) \cdot c_k \cdot x^{k-2} + \sum_{k=0}^{\infty} c_k \cdot x^k = 0$$

samma serie!

Sätt: $k-2=i$, $k=i+2$

sen: byt sen i mot k.

ändra i index så att x-termerna blir lika.

$$\sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2}\, x^k + \sum_{k=0}^{\infty} c_k \cdot x^k = 0$$

$$\sum_{k=0}^{\infty} \left((k+2)(k+1) c_{k+2} + c_k \right) x^k = 0$$

$k=0$: x^0: $(0+2)(0+1) \cdot c_2 + c_0 = 0 \iff c_2 = -\frac{c_0}{2}$

$k=1$: x^1: $3 \cdot 2 \cdot c_3 + c_1 = 0 \iff c_3 = -\frac{c_1}{3 \cdot 2}$

$k=2$: x^2: $4 \cdot 3 \cdot c_4 + c_2 = 0 \iff c_4 = -\frac{c_2}{4 \cdot 3} = +\frac{c_0}{4 \cdot 3 \cdot 2}$

$\vdots$

$$y(0) = c_0 + c_1 x + c_2 x^2 + \dots$$

$$= 1 + 0 - \frac{1}{2} x^2 + 0 + \frac{1}{4 \cdot 3 \cdot 2} x^4 + \dots$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \mathcal{O}(x^6)$$

$$= \cos x.$$