$\boxed{F20}$ • Inverstransformering
• Laplace transform av produkter

funktion: Laplace transform

$f(t) \sim \mathcal{L}\{f(t)\} = \int_0^\infty f(t) \cdot e^{-st} dt = F(s)$

$f(x) \quad \mathcal{L}\{f(x)\} = \int_0^\infty f(x) \cdot e^{-sx} dx = F(s)$

samma transform

$y(t) \quad \mathcal{L}\{y(t)\} = Y(s)$

$g(t) \quad G(s)$

$1 \quad \mathcal{L}\{1\} = \frac{1}{s} \quad \frac{0!}{s^1}$

konstant $-\pi \quad \mathcal{L}\{-\pi\} = -\pi \cdot \mathcal{L}\{1\} = -\pi \cdot \frac{1}{s}$

$e^{-at} \sim \frac{1}{s+a}$

$\sin(at) \sim \frac{a}{s^2+a^2}$

$\cos(at) \sim \frac{s}{s^2+a^2}$

$t \sim \frac{1!}{s^2}$

$t^2 \sim \frac{2!}{s^3}$

$t^3 \sim \frac{3!}{s^4}$

L2 $\quad a f(t) + b \cdot g(t) \sim a F(s) + b \cdot G(s)$

ex) $f(t) = (2t+1)^2 + \cos 2t - \sin\sqrt{3}t + e^{-\pi t}$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{4t^2 + 4t + 1\} + \mathcal{L}\{\cos 2t\} - \mathcal{L}\{\sin\sqrt{3}t\} + \mathcal{L}\{e^{-\pi t}\}$

$$4\cdot\frac{2}{s^3} + 4\cdot\frac{1}{s^2} + \frac{1}{s} \qquad \frac{s}{s^2+4} \quad - \quad \frac{\sqrt{3}}{s^2+3} \quad + \quad \frac{1}{s+\pi}$$

$$\mathcal{L}\{f(t)\} = F(s) = \frac{8}{s^3} + \frac{4}{s^2} + \frac{1}{s} + \frac{s}{s^2+4} - \frac{\sqrt{3}}{s^2+3} + \frac{1}{s+\pi}$$

ex) $\cosh(3t) = \dfrac{e^{3t} + e^{-3t}}{2}$ $\qquad \sinh(t) = \dfrac{e^{t} - e^{-t}}{2}$

$$\mathcal{L}\{\cosh(3t)\} = \frac{1}{2}\mathcal{L}\{e^{3t} + e^{-3t}\} =$$

$$= \frac{1}{2}\left(\frac{1}{s-3} + \frac{1}{s+3}\right). \quad = \frac{1}{2}\left(\frac{s+3+s-3}{(s-3)(s+3)}\right)$$

$$= \frac{1}{2}\left(\frac{2s}{s^2-9}\right) = \underline{\frac{s}{s^2-9}}.$$

$$F(s) = \frac{s}{s^2-9} = \frac{s}{(s+3)(s-3)} = \frac{A}{s+3} + \frac{B}{s-3}$$

faktorisering partialbråksuppdel.

Inverstransformering

tid-sida transformsida

$$\begin{tikzcd} \boxed{\text{de}} \arrow[r, "\mathcal{L}", "1"'] & \boxed{Y(s)} \arrow[d, "2"] \\ \boxed{y(t)} & \boxed{} \arrow[l, "\mathcal{L}^{-1}"', "3"] \end{tikzcd}$$

kvadratkompl. el faktorisering
partialbråksuppdelning

3 Inverstransf. är entydig (1-1) och linjär

$$\mathcal{L}^{-1}\{c\cdot F(s) + d\cdot G(s)\} =$$
$$= c\cdot \underbrace{\mathcal{L}^{-1}\{F(s)\}} + d\cdot \underbrace{\mathcal{L}^{-1}\{G(s)\}}$$
$$= c \quad f(t) \quad + d\cdot \quad g(t).$$

ex) $\mathcal{L}^{-1}\left\{\frac{8}{s^3}\right\} = \mathcal{L}^{-1}\left\{4\cdot \underbrace{\frac{2}{s^3}}_{\sim\, t^2}\right\} = 4\cdot t^2$

ex) $\mathcal{L}^{-1}\left\{\frac{5}{s^2+2}\right\} = \mathcal{L}^{-1}\left\{\frac{5}{\sqrt{2}}\cdot \underbrace{\frac{\sqrt{2}}{s^2+\sqrt{2}^2}}_{\sim\, \sin(\sqrt{2}t)}\right\} = \frac{5}{\sqrt{2}}\sin\sqrt{2}t.$

ex) $\mathcal{L}^{-1}\left\{\frac{1}{1-2s}\right\} = \mathcal{L}^{-1}\left\{\frac{-1}{2s-1}\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{2}\cdot\frac{1}{s-1/2}\right\}$

$= -\frac{1}{2}\, e^{1/2\cdot t}$

Ex) $F(s)=\dfrac{s+5}{(s+1)(s^2+1)}$ Bestäm $f(t)$.

ansats Partialbråksuppdelning

$$\frac{s+5}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1} = \frac{A(s^2+1)+Bs(s+1)+C(s+1)}{(s+1)(s^2+1)}$$

$$s+5 = A(s^2+1) + B(s^2+s) + C(s+1)$$

Identifiera koeff. i täljarna.

$$\left.\begin{aligned} s^2&: & 0 &= A+B \\ s&: & 1 &= B+C \\ s^0&: & 5 &= A+C \end{aligned}\right\}$$

$$\left(\begin{array}{ccc|c} 1&1&0&0 \\ 0&1&1&1 \\ 1&0&1&5 \end{array}\right) (-1) \sim \left(\begin{array}{ccc|c} 1&1&0&0 \\ 0&1&1&1 \\ 0&-1&1&5 \end{array}\right) (1)$$

$$\begin{array}{c} A\ B\ C \\ \left(\begin{array}{ccc|c} 1&1&0&0 \\ 0&1&1&1 \\ 0&0&2&6 \end{array}\right) \end{array}$$

$2C=6$

$\underline{C=3}$

$\underline{B}=1-3=\underline{-2}$

$\underline{A}=-B=\underline{2}$

$$F(s) = \frac{2}{s+1} + \frac{-2s+3}{s^2+1} =$$

$$= 2\cdot\frac{1}{s+1} - 2\cdot\frac{s}{s^2+1} + 3\cdot\frac{1}{s^2+1}$$

$$\sim \quad e^{-t} \qquad \sim \quad \cos t \qquad \sim \quad \sin t$$

$$\mathcal{L}^{-1}\{F(s)\} = \underline{f(t) = 2e^{-t} - 2\cos t + 3\sin t}$$

$$(1-x)\,y' = 2y$$

Sätt: $y = \sum_{k=0}^{\infty} c_k \cdot x^k = c_0 + c_1 x + c_2 x^2 + \dots$

$y' = \sum_{k=1}^{\infty} c_k \cdot k \cdot x^{k-1} = c_1 + 2c_2 x + \dots$

$$1 \cdot \underbrace{\sum_{1}^{\infty} c_k \cdot k \cdot x^{k-1}}_{y'} - x \cdot \underbrace{\sum_{1}^{\infty} c_k \cdot k \cdot x^{k-1}}_{y'} = 2 \cdot \sum_{k=0}^{\infty} c_k \cdot x^k$$

$$\sum_{k=1}^{\infty} c_k \cdot k \cdot x^{k-1} - \sum_{k=1}^{\infty} c_k \cdot k \cdot x^{k} = \cancel{2} \cdot \sum_{k=0}^{\infty} 2c_k \cdot x^k$$

✓ indexbyte till samma x-termer.

$$\sum_{k=0}^{\infty} c_{k+1}(k+1)\, x^k - \sum_{k=1}^{\infty} c_k\, k \cdot x^k = \sum_{k=0}^{\infty} 2c_k \cdot x^k$$

$k=0$: $c_1 \cdot 1 \cdot x^0 = 2c_0 \iff \underline{\underline{c_1 = 2c_0}}$

$k \geq 1$: $\sum_{k=1}^{\infty} \left[c_{k+1}(k+1) - c_k \cdot k - 2c_k \right] x^k = 0$

$k=1$: $c_2 \cdot 2 - c_1 \cdot 1 - 2c_1 = 0 \quad c_2 = \frac{3}{2} c_1 = 3c_0$

$k=2$: $c_3 \cdot 3 - c_2 \cdot 2 - 2c_2 = 0 \quad c_3 = \frac{4}{3} c_2 = 4c_0$

$\vdots$

$$y(x) = c_0 + 2c_0 x + 3c_0 x^2 + \dots =$$
$$= c_0 (1 + 2x + 3x^2 + 4x^3 + \dots = \underline{\underline{c_0 \cdot \sum_{k=0}^{\infty} (k+1)\, x^k}}$$

där $c_0 = y(0)$

Laplacetransform av produkter:

- $\boxed{\text{Produkt med } t}$: $t \cdot f(t)$

L5 $\boxed{\mathcal{L}\{t \cdot f(t)\} = -\frac{d}{ds}\left(\mathcal{L}\{f(t)\}\right) = -F'(s)}$

Laplacetransf. bara $f(t)$ (som är mult. med t)
Derivera sedan transformen m.a.p. s,
och byt tecken.

ex) $\mathcal{L}\{t \cdot 1\} = -\frac{d}{ds}(\mathcal{L}\{1\}) = -\frac{d}{ds}\left(\frac{1}{s}\right) = -\frac{d}{ds}(s^{-1}) = -(-1 \cdot s^{-2}) = \frac{1}{s^2}$.

ex) $\mathcal{L}\{t \cdot t\} = -\frac{d}{ds}\mathcal{L}\{t\} = -\frac{d}{ds}\left(\frac{1}{s^2}\right) = -\frac{d}{ds}(s^{-2}) = -(-2) \cdot s^{-3} = \frac{2}{s^3}$

ex) $\mathcal{L}\{t \cdot e^{-at}\} = -\frac{d}{ds}\mathcal{L}\{e^{-at}\} = -\frac{d}{ds}\left(\frac{1}{s+a}\right) = -\frac{d}{ds}\left((s+a)^{-1}\right) = -(-1) \cdot (s+a)^{-2} = \frac{1}{(s+a)^2}$

se även L21.

ex) $\mathcal{L}\{t^2 \cdot f(t)\} = (-1)(-1) \cdot \frac{d^2}{ds^2}\mathcal{L}\{f(t)\} = F''(s)$.

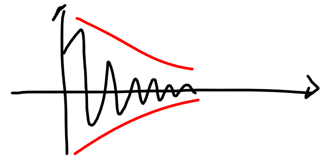
där $F(s) = \mathcal{L}\{f(t)\}$

ex) $\mathcal{L}\{t^2 \cdot 1\} = (-1)(-1) \cdot \frac{d^2}{ds^2}\left(\frac{1}{s}\right) = \frac{d^2}{ds^2}(s^{-1}) = \frac{d}{ds}(-1 \cdot s^{-2}) = (-1)(-2) \cdot s^{-3} = \frac{2}{s^3}$

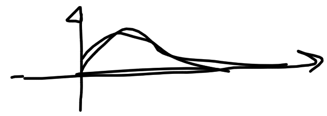
- $\boxed{\text{Produkt med } e^{-at}}$

dvs multiplikation med en <u>dämpande</u> faktor.

ex) $\underbrace{e^{-at}}_{\text{amplitud}} \cdot \sin \omega t$



ex) $e^{-at} \cdot t$



$$\mathcal{L}\{e^{-at} \cdot f(t)\} = \int_0^\infty e^{-at} \cdot f(t) \cdot e^{-st} \, dt =$$

$$= \int_0^\infty f(t) \cdot e^{-(s+a)t} \, dt = F(s+a) =$$

$$= F(s)\Big|_{s \to s+a}$$

Laplacetransform av $f(t)$.
Skifta sedan på transformsidan (translatera)
(med ombytt tecken på a ...)

ex) $\mathcal{L}\{e^{-at} \cdot 1\} = \mathcal{L}\{1\}\Big|_{s \to s+a} = \frac{1}{s}\Big|_{s \to s+a} = \underline{\underline{\frac{1}{s+a}}}$

ex) $\mathcal{L}\{e^{-2t} \cdot t\} = \mathcal{L}\{t\}\Big|_{s \to s+2} = \frac{1}{s^2}\Big|_{s \to s+2} = \underline{\underline{\frac{1}{(s+2)^2}}}$

ex) $\mathcal{L}\{e^{-3t} \cdot \cos 2t\} = \mathcal{L}\{\cos 2t\}\Big|_{s \to s+3} = \frac{s}{s^2+4}\Big|_{s \to s+3}$

$$= \underbrace{\frac{s+3}{(s+3)^2+4}} = \frac{s+3}{s^2+6s+13}$$

kvadratkomplettering el. faktorisering.

$$= \frac{s}{\underset{\sim \cos 2t}{s^2+4}}\Big|_{\underset{\sim e^{-3t}}{s \to s+3}}$$

Test

13 d) $D(s) = \dfrac{s+3}{s^2+4s+6}$

~~faktorisere~~
~~$s^2+4s+6=0$~~
~~$s=-2\pm\sqrt{4-6}$~~

$$= \frac{s+3}{(s+2)^2+2} =$$

$$= \frac{(s+2)+1}{(s+2)^2+(\sqrt{2})^2} = \frac{s+2}{(s+2)^2+\sqrt{2}^2} + \frac{1}{\sqrt{2}}\frac{\sqrt{2}\,1}{(s+2)^2+\sqrt{2}^2}$$

$$= \frac{s}{s^2+\sqrt{2}^2}\Big|_{s\to s+2} + \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{s^2+\sqrt{2}^2}\Big|_{s\to s+2}$$

$\sim$ $\cos\sqrt{2}t$ $\sim$ e^{-2t} $\sim$ $\sin\sqrt{2}t$ $\sim$ e^{-2t}

$$d(t) = e^{-2t}\cdot\cos\sqrt{2}t + \frac{1}{\sqrt{2}}e^{-2t}\cdot\sin\sqrt{2}t.$$

ex) $Y(s) = \dfrac{3}{(s-1)^2+2^2}$ Inverstransf ?

$$= \frac{3}{2}\cdot\frac{1\cdot 2}{s^2+2^2}\Big|_{s\to s-1}$$

$\sim$ $\sin 2t$ $\sim$ e^t

$$y(t) = \frac{3}{2}\cdot\sin 2t\cdot e^t$$