Laplacetransformen av derivator.

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\cdot e^{-st}\,dt = F(s)$$

$$\mathcal{L}\{y(t)\} = \int_0^\infty y(t)\cdot e^{-st}\,dt = Y(s)$$

$$\mathcal{L}\{y'(t)\} = \int_0^\infty \overset{\text{int}}{y'(t)}\cdot \overset{\text{der}}{e^{-st}}\,dt = \quad \text{PI= Partiell Integration}$$

$$= \left[y(t)\cdot e^{-st}\right]_0^\infty - \int_0^\infty y(t)\cdot e^{-st}(-s)\,dt$$

(där $y(t)$ växer högst exponentiellt.)

$$= 0 - y(0)\cdot e^0 + s\cdot \underbrace{\int_0^\infty y(t)\, e^{-st}\,dt}$$

$$= -y(0) + s\cdot Y(s)$$

$$\boxed{\therefore\ \mathcal{L}\{y'(t)\} = s\cdot Y(s) - y(0)}$$

b.v.

$$\mathcal{L}\{y''(t)\} = \int_0^\infty \overset{\text{int}}{y''(t)}\cdot \overset{\text{der}}{e^{-st}}\,dt \overset{PI}{=} \left[y'(t)\cdot e^{-st}\right]_0^\infty - \int_0^\infty y'(t)\cdot(-s)e^{-st}\,dt$$

$$= 0 - y'(0)\cdot e^0 + s\cdot\int_0^\infty y'(t)\cdot e^{-st}\,dt$$

$$= -y'(0) + s\cdot\left(s\cdot Y(s) - y(0)\right) =$$

$$\boxed{\mathcal{L}\{y''(t)\} = s^2\cdot Y(s) - s\cdot y(0) - y'(0)}$$

b.v.

p.s.s.

$$\mathcal{L}\{y'''(t)\} = s^3\cdot Y(s) - s^2\cdot y(0) - s\cdot y'(0) - y''(0)$$

$$\mathcal{L}\{y^{(4)}(t)\} = s^4\cdot Y(s) - s^3\cdot y(0) - s^2\cdot y'(0) - s\cdot y''(0) - y'''(0)$$

b.v

Ex) $\begin{cases} y'(t) - y(t) = e^{2t} \\ y(0) = 2 \end{cases}$ (1:a ordn. d.e. linjär)

$\mathcal{L}\{y(t)\} = Y(s)$

$\mathcal{L}\{y'(t)\} = s \cdot Y(s) - \underbrace{y(0)}_{=2} = s \cdot Y(s) - 2$

$\mathcal{L}\{e^{2t}\} = \frac{1}{s-2}$

Laplacetransf. av d.e.:

VL $\mathcal{L}\{y'(t) - y(t)\} = \mathcal{L}\{y'(t)\} - \mathcal{L}\{y(t)\} = \mathcal{L}\{e^{2t}\}$ HL

$(s \cdot Y(s) - 2) - Y(s) = \frac{1}{s-2}$ Lös ut $Y(s)$

$(s-1) \cdot Y(s) = \frac{1}{s-2} + \frac{2 \cdot (s-2)}{s-2} = \frac{2s-4+1}{s-2}$

$Y(s) = \frac{2s-3}{(s-2)(s-1)}$

Partialbråksuppdelning (PB)

Ansats

$Y(s) = \frac{2s-3}{(s-2)(s-1)} = \frac{A}{s-2} + \frac{B}{s-1}$

samma nämnare som i VL

ger täljarna:

$2s-3 = A(s-1) + B(s-2)$

identifiera

$$\begin{array}{lrcl} s: & 2 & = & A + B \\ s^0: & -3 & = & -A - 2B \\ \hline & -1 & = & 0 - B \end{array} \quad \Leftrightarrow \quad \underline{B=1}$$

$\underline{A=1}$

Ins i Ansatsen till PB

$\therefore Y(s) = \frac{1}{s-2} + \frac{1}{s-1}$

$\sim e^{2t}$ $\sim e^{t}$

inverstransf. ger

Svar: $\underline{\underline{y(t) = e^{2t} + e^{t}}}$

ex) $\begin{cases} y''(t) - 3y'(t) + 2y(t) = e^t \\ y(0) = 1 \\ y'(0) = 2 \end{cases}$

$$\mathcal{L}\{y(t)\} = Y(s) = Y$$
$$\mathcal{L}\{y'(t)\} = s \cdot Y(s) - y(0) = s \cdot Y(s) - 1$$
$$\mathcal{L}\{y''(t)\} = s^2 \cdot Y(s) - s \cdot y(0) - y'(0) = \underline{s^2 \cdot Y(s) - s - 2}$$
$$\mathcal{L}\{e^t\} = \frac{1}{s-1}$$

$$(s^2 \cdot \underline{Y(s)} - s - 2) - 3(s\,\underline{Y(s)} - 1) + 2 \cdot \underline{Y(s)} = \frac{1}{s-1}$$

Lös ut $Y(s)$

$$(s^2 - 3s + 2) \cdot Y(s) \; \underbrace{- s - 2 + 3}_{1} = \frac{1}{s-1}$$

$$\underbrace{(s^2 - 3s + 2)}_{s^2 - 3s + 2 = 0} \cdot Y(s) = \frac{1}{s-1} + \frac{(s-1)(s-1)}{(s-1)} = \frac{1 + (s-1)^2}{s-1}$$

$s = \frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{2 \cdot 4}{4}}$

$s = \frac{3 \pm 1}{2} < \begin{matrix} 2 \\ 1 \end{matrix}$

$$Y(s) = \frac{1 + s^2 - 2s + 1}{(s-1) \cdot (s^2 - 3s + 2)}$$

prova faktorisera.

$$Y(s) = \frac{s^2 - 2s + 2}{(s-1)(s-2)(s-1)} = \frac{s^2 - 2s + 2}{(s-1)^2 (s-2)}$$

Partialbråksuppdeln. Ansats:

$$Y(s) = \frac{s^2 - 2s + 2}{(s-1)^2 (s-2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s-2}$$

$$s^2 - 2s + 2 = A\underbrace{(s-1)(s-2)}_{s^2 - 3s + 2} + B(s-2) + C\underbrace{(s-1)^2}_{s^2 - 2s + 1}$$

id.

$$\left.\begin{aligned} s^2 &: & 1 &= A + C \\ s &: & -2 &= -3A + B - 2C \\ s^0 &: & 2 &= 2A - 2B + C \end{aligned}\right\} \quad \left(\begin{matrix} 1 & 0 & 1 \\ -3 & 1 & -2 \\ 2 & -2 & 1 \end{matrix} \left| \begin{matrix} 1 \\ -2 \\ 2 \end{matrix} \right.\right) \begin{matrix} 3 & -2 \end{matrix}$$

$$\sim \left(\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & -1 \end{matrix} \left| \begin{matrix} 1 \\ 1 \\ 0 \end{matrix} \right.\right) 2 \sim \left(\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix} \left| \begin{matrix} 1 \\ 1 \\ 2 \end{matrix} \right.\right) -1 \sim \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \left| \begin{matrix} -1 \\ -1 \\ 2 \end{matrix} \right.\right)$$

$A = -1$, $B = -1$, $C = 2$

forts.

$$Y(s) = \frac{s^2-2s+2}{(s-1)^2(s-2)} = \frac{-1}{s-1} + \frac{-1}{(s-1)^2} + \frac{2}{s-2}$$

$$\sim \quad -e^t \qquad \sim \quad -t\cdot e^t \qquad \sim \quad 2\cdot e^{2t}$$

inverstransf.

$$y(t) = -e^t - t\cdot e^t + 2e^{2t}$$

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ex) $\dfrac{dx}{dt} + 4\,x(t) = \sin 2t$ b.v. $x(0)=0$

$$\dot{x} + 4x = \sin 2t$$

$$\mathcal{L}\{x\} = X(s)$$

$$\mathcal{L}\{\dot{x}\} = s\cdot X(s) - x(0) = s\cdot X(s)$$

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+2^2}$$

- .. - . . . -

$$s\cdot X(s) + 4\,X(s) = \frac{2}{s^2+4}$$

$$(s+4)\cdot X(s) = \frac{2}{s^2+4}$$

$$X(s) = \frac{2}{(s+4)(s^2+4)}$$

Partialbråksuppd.

$$X(s) = \frac{2}{(s+4)(s^2+4)} = \frac{A}{s+4} + \frac{Bs+C}{s^2+4}$$

$\cancel{s^2+4=0}$ $\cancel{s^2=-4}$

$$2 = A(s^2+4) + Bs(s+4) + C(s+4)$$

s^2: $0 = A + B$

s: $0 = 0 + 4B + C$

s^0: $2 = 4A + 0 + 4C$

$\therefore$ $A = \frac{1}{10}$, $B = -\frac{1}{10}$, $C = \frac{4}{10}$

$$X(s) = \frac{1/10}{s+4} + \frac{-\frac{1}{10}s + 4/10}{s^2+4} = \frac{1}{10}\left[\frac{1}{s+4} - \frac{s}{s^2+4} + 2\cdot\frac{\cancel{4}\,2}{s^2+4}\right]$$

$$\sim \; e^{-4t} \qquad \sim \; -\cos 2t \qquad \sim \; 2\cdot\sin 2t$$

$$x(t) = \frac{1}{10}\left(e^{-4t} - \cos 2t + 2\sin 2t\right)$$