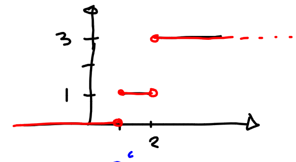
$\boxed{F24}$ Heaviside-, Ramp-, Impuls funktion
System av diff. ekv.

ex) $$f(t) = \begin{cases} 0 & t<1 \\ 1 & 1 \le t < 2 \\ 3 & t \geq 2 \end{cases}$$



$$f(t) = 1 \cdot (\underbrace{\overset{\text{på}}{\theta(t-1)}}_{=1 \text{ då } t>1} - \underbrace{\overset{\text{av}}{\theta(t-2)}}_{=1 \text{ då } t>2}) + 3 \cdot (\overset{\text{på}}{\theta(t-2)})$$

$$= \theta(t-1) \cdot 1 + \theta(t-2)\,(\overset{2}{\cancel{-1+3}})$$

$$\mathcal{L}\{f(t)\} = e^{-s} \cdot \mathcal{L}\{1\} + e^{-2s} \cdot \mathcal{L}\{2\}$$

$$= e^{-s} \cdot \frac{1}{s} + e^{-2s} \cdot 2 \cdot \frac{1}{s}.$$

T. Test. 19a) $y'' + y = \theta(t)$ bv. $y(0^-) = y'(0^-) = 0$

$$\mathcal{L}\{y\} = Y(s) = Y$$
$$\mathcal{L}\{y''\} = s^2 \cdot Y - s \cdot \underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=0}$$
$$\mathcal{L}\{\theta(t)\} = \mathcal{L}\{1\} = \frac{1}{s}$$

$$\theta(t) = \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$$

$$\theta(t-a) = \begin{cases} 1 & t>a \\ 0 & t<a \end{cases}$$

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$$s^2 \cdot Y + Y = \frac{1}{s}$$

$$(s^2+1) \cdot Y = \frac{1}{s}$$

$$Y = \frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$1 = A(s^2+1) + Bs^2 + Cs$$

$$\left.\begin{array}{lll} s^2: & 0 = & A + B \\ s: & 0 = & C \\ s^0: & 1 = & A \end{array}\right\} \quad \begin{array}{l} B = -1 \\ C = 0 \\ A = 1 \end{array}$$

$$Y = \frac{1}{s} - \frac{s}{s^2+1}$$

inverstransf. ~ 1 $\sim \cos t$

Svar: $y(t) = (1 - \cos t)\,\theta(t)$

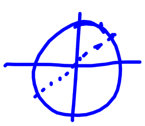
Ex) $f(t) = \begin{cases} \sin \pi t & 1 \le t \le 2 \\ 0 & \text{f.ö.} \end{cases}$

$$f(t) = \sin \pi t \cdot [\underset{t \geq 1}{\overset{på}{\theta(t-1)}} - \underset{t \geq 2}{\overset{av}{\theta(t-2)}}]$$

$$\boxed{L3: \ \mathcal{L}(f(t-a)\cdot\theta(t-a)) = e^{-as}\cdot\mathcal{L}\{f(t)\}}$$

$$f(t) = \theta(t-1)\cdot \sin \pi(t-1+1) - \theta(t-2)\cdot \sin \pi(t-2+2)$$

$$= \theta(t-1)\cdot \underbrace{\sin[\pi(t-1)+\pi]}_{= -\sin \pi(t-1)} - \theta(t-2)\cdot \underbrace{\sin[\pi(t-2)+2\pi]}_{\cdot \sin \pi(t-2)}$$



$\sin(\alpha+\pi) =$
$= -\sin\alpha$

ofördröjd transf.

ofördröjd transf

Laplacetransf.

$$\mathcal{L}\{f(t)\} = e^{-s}\cdot \mathcal{L}\{-\sin \pi \cdot t\} - e^{-2s}\cdot \mathcal{L}\{\sin \pi t\}$$

$$= -e^{-s}\left(\frac{\pi}{s^2+\pi^2}\right) - e^{-2s}\left(\frac{\pi}{s^2+\pi^2}\right)$$

$$= \frac{-\pi}{s^2+\pi^2}\left(e^{-s}+e^{-2s}\right)$$

Ex) Inverstransf.

$$F(s) = \frac{e^{-6s}}{s^2+8s+15} = e^{-6s} \underbrace{\frac{1}{(s+3)(s+5)}}_{\text{jobba med detta först.}}$$

faktorisera el. kvadratkompl.

$s^2+8s+15=0$

$s=-4\pm\sqrt{16-15}$

$s=-4\pm 1 \; \begin{cases} -3 \\ -5 \end{cases}$

$$\frac{1}{(s+3)(s+5)} = \frac{A}{s+3} + \frac{B}{s+5} = \frac{1/2}{s+3} + \frac{-1/2}{s+5}$$

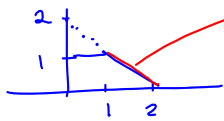
$$F(s) = e^{-6s}\left(\frac{1}{2}\cdot\frac{1}{s+3} - \frac{1}{2}\cdot\frac{1}{s+5}\right)$$

fördröjd till $t=6$ $\quad \sim e^{-3t} \quad \sim e^{-5t}$

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \theta(t-6)\cdot\left(\frac{1}{2}e^{-3(t-6)} - \frac{1}{2}e^{-5(t-6)}\right)$$

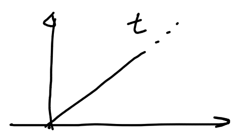
$$f(t) = 3\cdot\left(\theta(t-1)-\theta(t-2)\right) + 1\cdot\,\theta(t-2)$$

$$f(t) = 1\left(\theta(t)-\theta(t-1)\right) + (2-t)\cdot\left(\theta(t-1)-\theta(t-2)\right)$$



Rampfunktion.

$$\rho(t) = \begin{cases} 0 & t<0 \\ t & t\geq 0 \end{cases}$$



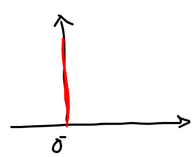
t.ex linjärt ökande spänning i elektriskt system.

$$\rho(t) = t\cdot\theta(t)$$

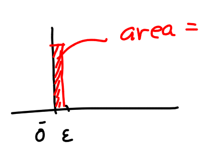
$$\boxed{\mathcal{L}\{\rho(t)\} = \mathcal{L}\{t\cdot\theta(t)\} = \mathcal{L}\{t\} = \frac{1}{s^2}}$$

Impulsfunktion "Dirac-delta"

$$\delta(t) = \begin{cases} \text{"}\infty\text{"} & t=0 \\ 0 & t\neq 0 \end{cases}$$



$$\delta(t) = \begin{cases} \lim\limits_{\varepsilon\to 0} \frac{1}{\varepsilon} & 0^- < t < \varepsilon \\ 0 & \text{f.ö.} \end{cases}$$



$\delta(t-a)$ ger en stöt vid $t=a$.

- $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

  $\int_{0^-}^{\infty} \delta(t)\,dt = 1$

- $\int_{0^-}^{\infty} f(t)\cdot\delta(t-a)\,dt = f(a)$

Dirac-delta stöter ut $f(t)$ då $t=a$

$$\boxed{\mathcal{L}\{\delta(t)\} = 1}$$

System av differentialekvationer

Ex) $\begin{cases} \frac{dx}{dt} + 3x + 5y = e^{-t} \\ \frac{dy}{dt} - 3y - 5x = 0 \end{cases}$ $\quad x(0)=0,\ y(0)=0$

Laplacetransf. $\mathcal{L}\{x(t)\} = X(s) = X$

$\mathcal{L}\{y(t)\} = Y(s) = Y$

$\mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$

$$\begin{cases} (s\cdot X - \underbrace{x(0)}_{=0}) + 3\cdot X + 5\cdot Y = \frac{1}{s+1} \\ (s\cdot Y - \underbrace{y(0)}_{=0}) - 3\cdot Y - 5\cdot X = 0 \end{cases}$$

$$\begin{cases} (s+3)\cdot X + 5\cdot Y = \frac{1}{s+1} \\ -5\,X + (s-3)\cdot Y = 0 \end{cases}$$

$$\boxed{\underbrace{\begin{pmatrix} s+3 & 5 \\ -5 & s-3 \end{pmatrix}}_{A} \cdot \begin{pmatrix} X \\ Y \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{s+1} \\ 0 \end{pmatrix}}_{b}}$$

i Matrisform.

Cramersregel - fungerar då det finns entydig lösn. till kvadratiska ekv. syst.

$$X = \frac{\begin{vmatrix} \frac{1}{s+1} & 5 \\ 0 & s-3 \end{vmatrix}}{\begin{vmatrix} s+3 & 5 \\ -5 & s-3 \end{vmatrix}} = \frac{\det A_1(b)}{\det A} = \frac{\frac{1}{s+1}(s-3) - 0\cdot 5}{(s+3)(s-3) - (-5)\cdot 5} =$$

ansats PB

$$= \frac{s-3}{s+1\cdot(s^2-9+25)} = \frac{s-3}{(s+1)(s^2+16)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+16}$$

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$$s-3 = A(s^2+16) + Bs(s+1) + C(s+1)$$

$$\left.\begin{array}{lrl} s^2: & 0 = & A + B \\ s: & 1 = & B + C \\ s^0: & -3 = & 16A + C \end{array}\right\} \cdots \quad \begin{array}{l} A = -\frac{4}{17} \\ B = \frac{4}{17} \\ C = \frac{13}{17} \end{array}$$

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$$X = -\frac{4}{17}\cdot\frac{1}{s+1} + \frac{\frac{4}{17}s - \frac{13}{17}}{s^2+16}$$

$$= -\frac{4}{17}\cdot\underbrace{\frac{1}{s+1}}_{\sim\, e^{-t}} + \frac{4}{17}\cdot\underbrace{\frac{s}{s^2+4^2}}_{\sim\, \cos 4t} - \frac{13}{17}\cdot\frac{1}{4}\underbrace{\frac{4}{s^2+16}}_{\sim\, \sin 4t}$$

$$x(t) = -\frac{4}{17}\cdot e^{-t} + \frac{4}{17}\cos 4t - \frac{13}{68}\sin 4t.$$

$$Y = \frac{\det A_2(b)}{\det A} = \frac{\begin{vmatrix} s+3 & \frac{1}{s+1} \\ -5 & 0 \end{vmatrix}}{\begin{vmatrix} s+3 & 5 \\ -5 & s-3 \end{vmatrix}} = \frac{(s+3)\cdot 0 - (-5)\cdot\frac{1}{s+1}}{s^2+16} =$$

$$= \frac{5}{(s+1)(s^2+16)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+16}$$

$A = 5/17 = C$
$B = -5/17$

$$Y = \frac{5}{17}\cdot\frac{1}{s+1} - \frac{5}{17}\cdot\left(\frac{s-1}{s^2+16}\right)$$

$$= \frac{5}{17}\cdot\frac{1}{s+1} - \frac{5}{17}\left(\frac{s}{s^2+4^2} - \frac{1}{4}\cdot\frac{4}{s^2+4^2}\right)$$

e^{-t} $\cos 4t$ $\sin 4t$

$$y(t) = \frac{5}{17}\cdot e^{-t} - \frac{5}{17}\cos 4t + \frac{5}{68}\sin 4t$$