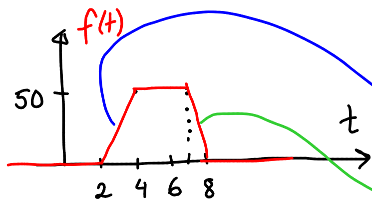
$\boxed{27\text{-}28}$ Integralekvation, Faltning

Rep.



Laplacetransf. $f(t)$.

$$f(t)=\begin{cases} 0 & t<2 \\ 25t-50 & 2\leq t<4 \\ 50 & 4\leq t<7 \\ -50t+400 & 7\leq t<8 \\ 0 & t\geqslant 8 \end{cases}$$

$y = kt+m$

$k = \frac{\Delta y}{\Delta t} = \frac{50-0}{4-2} = 25$

$0 = 25\cdot 2 + m \iff m=-50$

$\therefore\ y_1 = 25t-50$

$y = kt+m$

$k = \frac{\Delta y}{\Delta t} = \frac{50-0}{7-8} = -50$

$0 = -50\cdot 8 + m \iff m=+400$

$$f(t) = (25t-50)\cdot[\theta(t-2)-\theta(t-4)] + 50\cdot[\theta(t-4)-\theta(t-7)] +$$
$$+ (-50t+400)[\theta(t-7)-\theta(t-8)]. =$$

(dela upp i brytpunkter)

$$f(t) = \theta(t-2)\cdot[25(t-2)] +$$
$$+\ \theta(t-4)\cdot[\underbrace{-25t+50+50}_{-25(t-4)}] +$$
$$+\ \theta(t-7)\cdot[\underbrace{-50-50t+400}_{-50(t-7)}] +$$
$$+\ \theta(t-8)\cdot(\underbrace{50t-400}_{50(t-8)})$$

$$\mathcal{L}\{f(t)\} =$$
$$e^{-2s}\cdot\mathcal{L}\{25t\} +$$
$$+e^{-4s}\cdot\mathcal{L}\{-25t\} +$$
$$+e^{-7s}\cdot\mathcal{L}\{-50t\} +$$
$$+e^{-8s}\cdot\mathcal{L}\{50t\} =$$

$$\mathcal{L}\{f(t)\} = e^{-2s}\cdot\frac{25}{s^2} + e^{-4s}\left(\frac{-25}{s^2}\right) + e^{-7s}\cdot\left(\frac{-50}{s^2}\right) + e^{-8s}\cdot\frac{50}{s^2}$$

$$= \frac{25}{s^2}\cdot\left[e^{-2s} - e^{-4s} - 2e^{-7s} + 2e^{-8s}\right].$$

T. kap 5.1

Test 1)
$$\begin{cases} x' + x - y' + y = 0 \\ x' - x + y' + y = 0 \end{cases} \qquad \begin{matrix} x(0) = 3 \\ y(0) = 4 \end{matrix}$$

Laplace transf.
$$\mathcal{L}\{x\} = X(s) = X$$
$$\mathcal{L}\{y\} = Y$$
$$\mathcal{L}\{x'\} = s \cdot X - \underbrace{x(0)}_{=3} = sX - 3$$
$$\mathcal{L}\{y'\} = s \cdot Y - \underbrace{y(0)}_{4} = sY - 4$$

$$\begin{cases} (sX - 3) + X - (sY - 4) + Y = 0 \\ (sX - 3) - X + (sY - 4) + Y = 0 \end{cases}$$

$$\begin{cases} (s+1)X + (-s+1)Y = 3 - 4 \\ (s-1)X + (s+1)Y = 3 + 4 \end{cases}$$

$$\underbrace{\begin{pmatrix} (s+1) & (-s+1) \\ s-1 & s+1 \end{pmatrix}}_{A} \begin{pmatrix} X \\ Y \end{pmatrix} = \underbrace{\begin{pmatrix} -1 \\ 7 \end{pmatrix}}_{b}$$

$$X = \frac{\det A_1(b)}{\det A} = \frac{\begin{vmatrix} -1 & -(s-1) \\ 7 & s+1 \end{vmatrix}}{\begin{vmatrix} s+1 & -(s-1) \\ s-1 & s+1 \end{vmatrix}} = \frac{-(s+1) + 7(s-1)}{(s+1)^2 + (s-1)^2}$$

$$X = \frac{-s - 1 + 7s - 7}{s^2 + \cancel{2s} + 1 + s^2 - \cancel{2s} + 1} = \frac{6s - 8}{2(s^2+1)}$$

$$= 3 \underbrace{\frac{s}{s^2+1}}_{\sim \cos t} - 4 \underbrace{\frac{1}{s^2+1}}_{\sim \sin t}$$

$$\underline{x(t) = 3\cos t - 4 \sin t.}$$

$$Y = \frac{\det A_2(b)}{\det A} = \frac{\begin{vmatrix} s+1 & -1 \\ s-1 & 7 \end{vmatrix}}{2(s^2+1)} = \frac{7s+7-(s-1)(-1)}{2(s^2+1)}$$

$$Y = \frac{8s+6}{2(s^2+1)} = 4\,\underbrace{\frac{s}{s^2+1}}_{\sim \cos t} + 3\cdot\underbrace{\frac{1}{s^2+1}}_{\sim \sin t}$$

$$\underline{y(t) = 4\cos t + 3\sin t}$$

Laplacetransf. av integral

Härledning: $g(t) = \int_0^t f(x)\,dx$

Sök: $\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} =$

$\mathcal{L}\{g(t)\} = G(s).$

Derivera m.a.p t.

$$g'(t) = \frac{d}{dt}\left(\int_0^t f(x)\,dt\right\} = f(t)$$

Laplacetransf.

VL: $\mathcal{L}\{g'(t)\} = s\cdot G(s) - g(0)$

HL: $\mathcal{L}\{f(t)\} = F(s)$

$$\therefore \quad s\cdot G(s) - \underbrace{g(0)}_{?} = F(s)$$

$$g(0) = \int_0^0 f(x)\,dx = 0$$

$$s\cdot G(s) - 0 = F(s)$$

$$G(s) = \frac{1}{s}\cdot F(s)$$

$$\therefore \quad \mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \frac{1}{s}\cdot F(s) = \frac{1}{s}\cdot\mathcal{L}\{f(t)\}.$$

$$\frac{14s+2}{(s+1)^2+4} = \frac{14(s+1-1)}{(s+1)^2+4} + \frac{2}{(s+1)^2+4}$$

$$= 14 \cdot \frac{s+1}{(s+1)^2+2^2} \quad \overset{-12}{\cancel{-14+2}} \cdot \frac{1}{(s+1)+4}$$

$$= 14 \cdot \frac{s+1}{(s+1)^2+2^2} - 6 \cdot \frac{2}{(s+1)^2+2^2}$$

$$= 14 \cdot \frac{s}{s^2+2^2}\Big|_{s\to s+1} - 6 \cdot \frac{2}{s^2+2^2}\Big|_{s\to s+1}$$

$$\cos 2t \cdot \tilde{\bar{e}}^{t} \qquad \sin 2t \cdot \tilde{\bar{e}}^{t}$$

$$y(t) = 14 \cdot \cos 2t \cdot e^{-t} - 6 \sin 2t \cdot \bar{e}^{t}$$

Test.
22a) $$y(t) + \int_0^t y(x)\,dx = 1$$ integralekvation.

Sök: $y(t)$.

Laplacetransformera.

$$\mathcal{L}\{y(t)\} = Y(s) = Y$$

$$\mathcal{L}\{\int_0^t y(x)\,dx\} = \frac{1}{s}\cdot Y$$ $\boxed{L9}$

$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$Y + \frac{1}{s}\cdot Y = \frac{1}{s}$$ Lös ut Y

$$Y\left(1+\frac{1}{s}\right) = \frac{1}{s}$$

$$Y = \frac{1}{s\left(1+\frac{1}{s}\right)} = \frac{1}{s+1}$$

inverstransf. $$\underline{\underline{y(t) = e^{-t}}}$$

Faltning

	Mult. på tidssidan	Transformsidan (frekvenssidan)
L5	$t \cdot f(t)$	$-F'(s) = -\frac{d}{ds} \mathcal{L}\{f(t)\}$
L4	$e^{at} \cdot f(t)$	$F(s)_{s \to s-a} = F(s-a) = \mathcal{L}\{f(t)\}_{s \to s-a}$
	dämpning/förstärkn.	skiftning på transf. sidan

faltning (convolution)

Produkt på transf. sidan.

$f(t) * g(t) =$ $\qquad F(s) \cdot G(s)$

$= (f * g)(t) =$

"vikt funktion"

$$= \int_0^t f(t-x) \cdot g(x) \, dx =$$

"vikt funktion"

$$= \int_0^t f(x) \cdot g(t-x) \, dx$$

$$\mathcal{L}\{f * g(t)\} = \mathcal{L}\left\{\int_0^t f(t-x) \cdot g(x) \, dx\right\} = F(s) \cdot G(s) =$$

$$= \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}.$$

ex) $$f(t) = \underbrace{\int_0^t \sin x \cdot \cos(t-x)\,dx}_{(\sin * \cos)(t)}$$

Faltningen beror av t.

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\}\cdot\mathcal{L}\{\cos(t)\} =$$

$$= \frac{1}{s^2+1}\cdot\frac{s}{s^2+1} = \frac{s}{(s^2+1)^2}$$

$$= \frac{1}{2}\,\frac{2s}{(s^2+1)^2}$$

$\sim$

$t\cdot\sin t$ (L24)

$$f(t) = \frac{1}{2}\cdot t\cdot\sin t$$

$$\mathcal{L}\Big\{\underbrace{\int_0^t 1\cdot f(x)\,dx}_{1*f(t)}\Big\} = \mathcal{L}\{1\}\cdot\mathcal{L}\{f(t)\} =$$

$$= \frac{1}{s}\cdot F(s).$$

$$\mathcal{L}\Big\{\underbrace{\int_0^t 1\cdot y(t-x)\,dx}_{1*y(t)}\Big\} = \mathcal{L}\{1\}\cdot\mathcal{L}\{y(t)\} = \frac{1}{s}\cdot Y(s).$$

$$\mathcal{L}\left\{ \int_0^t \sin x \cdot \cos(x-t)\,dx \right\} =$$

$$= \mathcal{L}\left\{ \underbrace{\int_0^t \sin x \, \underbrace{\cos(-(t-x))}_{\cos(t-x)} \, dx}_{\sin(t) * \cos(t)} \right\}$$

$\cos(-\theta) = \cos\theta$

$$= \mathcal{L}\{\sin(t)\} \cdot \mathcal{L}\{\cos(t)\}$$

$$= \frac{1}{s^2+1} \cdot \frac{s}{s^2+1} =$$

Extraövningar till Laplacetransf. (finns i Fronter)

4. $f(t) = e^{at} * e^{bt}$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{at}\} \cdot \mathcal{L}\{e^{bt}\} =$$

$$= \frac{1}{s-a} \cdot \frac{1}{s-b} = \frac{1}{(s-a)(s-b)}$$

$$\frac{1}{(s-a)(s-b)} = \frac{A}{s-a} + \frac{B}{s-b} = \frac{A(s-b) + B(s-a)}{(s-a)(s-b)}$$

$$\left.\begin{array}{ll} s: & 0 = A + B \\ s^0: & 1 = -bA - aB \end{array}\right\} \quad \left(\begin{array}{cc|c} 1 & 1 & 0 \\ -b & -a & 1 \end{array}\right) b \sim$$

$$\sim \begin{array}{cc} A & B \\ \end{array}$$
$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & b-a & 1 \end{array}\right) \quad B = \frac{1}{b-a} \quad A = -B = -\frac{1}{b-a}.$$

$$\mathcal{L}\{f(t)\} = -\frac{1}{(b-a)} \cdot \underbrace{\frac{1}{(s-a)}}_{\sim\, e^{at}} + \frac{1}{(b-a)} \cdot \underbrace{\frac{1}{(s-b)}}_{\sim\, e^{bt}}$$

$$f(t) = -\frac{1}{b-a} \cdot e^{at} + \frac{1}{b-a} \cdot e^{bt} = e^{at} * e^{bt}.$$

$$= \int_0^t e^{ax} \cdot e^{b(t-x)}\,dx = \ldots$$

$$\frac{12}{s^7} = \frac{12}{6!} \cdot \frac{6!}{s^7} = \frac{\cancel{12}}{\cancel{6} \cdot 5 \cdot 4 \cdot 3 \cdot \cancel{2} \cdot 1} \cdot \frac{6!}{s^7}$$

$$\sim \frac{1}{60} \cdot t^6$$

$$y(t) = t + \underbrace{\int_0^t y(x)\cdot \sin(t-x)\,dx}_{y(t) * \sin(t)}$$

Faltning bare av t.

Laplacetransf.

$$Y = \frac{1}{s^2} + \mathcal{L}\{y\}\cdot\mathcal{L}\{\sin t\}$$

$$Y = \frac{1}{s^2} + Y\cdot\frac{1}{s^2+1}$$

Løs ut Y

$$Y\left(1 - \frac{1}{s^2+1}\right) = \frac{1}{s^2}$$

$$Y\left(\frac{s^2+\cancel{1}-\cancel{1}}{s^2+1}\right) = \frac{1}{s^2}$$

$$Y = \frac{1\,(s^2+1)}{s^2\cdot s^2} = \frac{1}{s^2} + \frac{1}{s^4}$$

$$\sim \quad t \qquad \frac{1}{3!}\cdot\frac{3!}{s^4}$$

$$\sim \quad \frac{1}{6}\cdot t^3$$

$$\underline{y(t) = t + \frac{1}{6}t^3}$$

$$\frac{+(3s+9)}{+(s^2+s+2)} = \frac{3s+9}{(s+\frac{1}{2})^2+\frac{7}{4}}$$

$$= \frac{3(s+\frac{1}{2}-\frac{1}{2})}{(s+\frac{1}{2})^2+\frac{7}{4}} + \frac{9}{(s+\frac{1}{2})^2+\frac{7}{4}}$$

$$= 3\cdot\frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2+\frac{7}{4}} + 2\cdot\frac{7.5}{\sqrt{7}}\cdot\frac{\frac{\sqrt{7}}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{7}}{2})^2}$$

$9-1.5=7.5$

$$= 3\left.\frac{s}{s^2+(\frac{\sqrt{7}}{2})^2}\right|_{s\to s+\frac{1}{2}} + \frac{15}{\sqrt{7}}\cdot\left.\frac{\sqrt{7}/2}{s^2+(\frac{\sqrt{7}}{2})^2}\right|_{s\to s+\frac{1}{2}}$$

$$3\cdot\cos\frac{\sqrt{7}}{2}t \quad e^{-\frac{1}{2}t} + \frac{15}{\sqrt{7}}\cdot\sin\frac{\sqrt{7}}{2}t \quad e^{-\frac{1}{2}t}$$