$\boxed{F29}$ Repetition

Cramers regel - sätt att lösa ekv. system med entydig lösning

ex) $\begin{cases} 3x - 2y = 6 \\ -5x + 4y = 8 \end{cases}$

$$\underbrace{\begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} 6 \\ 8 \end{pmatrix}}_{b}$$

$$x = \frac{\det A_1(b)}{\det A} = \frac{\begin{vmatrix} 6 & -2 \\ 8 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -5 & 4 \end{vmatrix}}$$

← första kol. i A byts mot till b.

$$x = \frac{6 \cdot 4 - 8(-2)}{3 \cdot 4 - (-5)(-2)} = \frac{24+16}{12-10} = \frac{40}{2} = 20$$

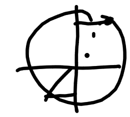
$$y = \frac{\det A_2(b)}{\det A} = \frac{\begin{vmatrix} 3 & 6 \\ -5 & 8 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -5 & 4 \end{vmatrix}} = \frac{3 \cdot 8 - (-5)6}{2} = \frac{24+30}{2} = 27$$

Svar: $\begin{cases} x = 20 \\ y = 27 \end{cases}$

Ex) $\begin{cases} y''(t) + 4y(t) = f(t) \\ y(0) = y'(0) = 0 \end{cases}$ $\quad f(t) = \begin{cases} \sin t & 0 \leq t \leq \pi \\ 0 & \text{f.ö.} \end{cases}$

Sök: $y\left(\frac{3\pi}{2}\right)$

Skriv om HL m.h.a. Heaviside



$$f(t) = \sin t\,(\theta(t) - \theta(t-\pi))$$

$$= \theta(t)\cdot \sin t + \theta(t-\pi)\cdot(-\sin t)$$

$$= \theta(t)\cdot \sin t + \theta(t-\pi)\cdot(-\sin(t-\pi+\pi))$$

halvt varv byt tecken på sin.

$$\underbrace{-\sin((t-\pi)+\pi)}_{=\sin(t-\pi)}$$

$$\sim e^{-0\cdot s}\frac{1}{s^2+1} \qquad \sim e^{-\pi s}$$

$\sim$ o-fördröjd

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$$

Laplacetransf.

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2+1} + e^{-\pi s}\cdot\frac{1}{s^2+1}.$$

$$\mathcal{L}\{y(t)\} = Y$$

$$\mathcal{L}\{y''(t)\} = s^2\cdot Y - s\cdot\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=0}$$

ins i d.e.

$$s^2\cdot Y + 4Y = \frac{1}{s^2+1}\left(1 + e^{-\pi s}\right)$$

$$s^2\cdot Y + 4Y = \frac{1}{s^2+1}\left(1+e^{-\pi s}\right)$$ Løs ut Y

$$(s^2+4)\cdot Y = \frac{1}{s^2+1}\left(1+e^{-\pi s}\right)$$

$$Y = \frac{(1+e^{-\pi s})\cdot 1}{(s^2+1)\cdot(s^2+4)}$$

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

.... ger $A = 0 = C$, $B = 1/3$, $D = -1/3$

$$Y = (1-e^{-\pi s})\cdot\left(\frac{1/3}{s^2+1} - \frac{1/3}{s^2+4}\right)$$

$$= \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{2}\cdot\frac{1\cdot 2}{s^2+4}\right) - \frac{1}{3}\, e^{-\pi s}\left(\frac{1}{s^2+1} - \frac{1}{2}\frac{1\cdot 2}{s^2+4}\right)$$

$\sim \sin t$ $\quad \sim \frac{1}{2}\sin 2t$ $\quad \sim \sin t$ $\quad -\frac{1}{2}\sin 2t$

inverstransf.

$$y(t) = \frac{1}{3}(\sin t) - \frac{1}{6}\sin 2t - \frac{1}{3}\,\theta(t-\pi)\cdot\left(\sin(t-\pi) - \frac{1}{2}\sin 2(t-\pi)\right)$$

Sjekk $y\left(\frac{3\pi}{2}\right) = \frac{1}{3}\underbrace{\sin\frac{3\pi}{2}}_{-1} - \frac{1}{6}\underbrace{\sin \cancel{2}\cdot\frac{3\pi}{\cancel{2}}}_{0} - \frac{1}{3}\underbrace{\theta\left(\frac{3\pi}{2}-\pi\right)}_{1}\left(\underbrace{\sin\left(\frac{3\pi}{2}-\pi\right)}_{1} - \frac{1}{2}\underbrace{\sin 2\left(\frac{3\pi}{2}-\pi\right)}_{0}\right)$

$$= -\frac{1}{3} - 0 - \frac{1}{3}(1-0) = \underline{\underline{-\frac{2}{3}}}$$

$$y(t) = \begin{cases} \frac{1}{3}\sin t - \frac{1}{6}\sin 2t & ;\ 0<t<\pi \\ \frac{1}{3}\sin t - \frac{1}{6}\sin 2t - \frac{1}{3}\sin(t-\pi) + \frac{1}{6}\sin 2(t-\pi) & ;\ t \geq \pi \end{cases}$$

Tenta maj 2015

1a) $z^2 - (2+2i)z + 5 - 10i = 0$ ej reell ekv.

kvadrat kompl.

$(z-(1+i))^2 - (1+i)^2 + 5 - 10i = 0$

$\underbrace{(z-(1+i))}_{=w}{}^2 - \underbrace{(\cancel{1}+2i+\cancel{i^2})+5-10i}_{=5-12i} = 0$

$w = z - 1+i$

$w^2 = -5 + 12i$

Sätt: $w = x+iy$ $x, y \in \mathbb{R}$

$\Rightarrow w^2 = (x+iy)^2 = x^2 + 2xyi + \underbrace{i^2y^2}_{-y^2}$

$x^2 + 2xyi - y^2 = -5 + 12i$

Re: $\begin{cases} x^2 - y^2 = -5 \\ 2xy = 12 \end{cases}$ Im: $\Rightarrow x = \frac{6}{y}$

$\frac{36}{y^2} - y^2 = -5$

$36 - y^4 = -5y^2 \Rightarrow y^4 - 5y^2 - 36 = 0$

$y^2 = \frac{5}{2} \pm \sqrt{\frac{25}{4} + 36}$

$y^2 = \frac{5 \pm 13}{2}$

$y^2 = 9$

$\cancel{y^2 = -\frac{8}{2}}$ ty $y \in \mathbb{R}$.

$y = \pm 3$

$x = \frac{6}{y} = \pm 2$

$\therefore w = x+iy = \pm(2+3i)$

$z - (1+i) = w$

$z = \pm(2+3i) + (1+i)$

$z_1 = 2+3i+1+i = 3+4i$

$z_2 = -2-3i+1+i = -1-2i$

Svar

1b) $z = \dfrac{2(1-i)}{1+i} e^{i\frac{\pi}{6}}$

så $|z| = \dfrac{|2| \cdot |1-i|}{|1+i|} \cdot \left|e^{i\frac{\pi}{6}}\right|$

$|1-i| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$|1+i| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\boxed{e^{i\theta} = \cos\theta + i \cdot \sin\theta}$

$\left|e^{i\frac{\pi}{6}}\right| = 1$ ty $\left|\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right| = \sqrt{\cos^2\frac{\pi}{6} + \sin^2\frac{\pi}{6}} =$

$= 1$ (trig. ettan).

$\therefore |z| = \dfrac{2 \cdot \sqrt{2}}{\sqrt{2}} \cdot 1 = \underline{\underline{2}}$

2. a) $y'' + 4y' + 3y = 2\sin x + 4\cos x$

homogen lösn.:

$$y_h = C\cdot e^{rx}$$
$$y_h' = Cr\cdot e^{rx}$$
$$y_h'' = Cr^2\cdot e^{rx}$$

$$C\cdot e^{rx}\cdot(\underbrace{r^2+4r+3}_{\text{karakteristisk ekv}}) = 0$$

$$r = -2 \pm \sqrt{4-3}$$
$$r = -2 \pm 1 \begin{cases} -1 \\ -3 \end{cases}$$

$$\therefore \underline{y_h = C_1\cdot e^{-x} + C_2\cdot e^{-3x}}$$

partikulär lösn.:

Ansatz: $y_p = A\cdot\sin x + B\cos x$
$y_p' = A\cos x - B\sin v$
$y_p'' = -A\sin x - B\cos x$ } ins i d.e.

$$-A\sin x - B\cos x + 4(A\cos x - B\sin x) + 3(A\sin x + B\cos x) =$$
$$= 2\sin x + 4\cos x$$

$$\begin{array}{l} \sin x: \\ \cos x: \end{array} \begin{cases} -A - B4 + 3A = 2 \\ -B + 4A + 3B = 4 \end{cases}$$

$$\begin{cases} 2A - 4B = 2 \\ 4A + 2B = 4 \end{cases}$$

$$A = \frac{\begin{vmatrix} 2 & -4 \\ 4 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -4 \\ 4 & 2 \end{vmatrix}} = \frac{2\cdot 2 - 4(-4)}{2\cdot 2 - 4(-4)} = \frac{20}{20} = 1$$

$$B = \frac{\begin{vmatrix} 2 & 2 \\ 4 & 4 \end{vmatrix}}{20} = \frac{8-8}{20} = 0.$$

ins i ansatsen till y_p.

$$\therefore \underline{y_p = \sin x}$$

AL. $y(x) = y_h + y_p = \underline{C_1 e^{-x} + C_2\cdot e^{-3x} + \sin x}$

2b) $(1+x)y' + y = 0$ $\qquad y(0) = 4$

$(1+x)\frac{dy}{dx} = -y$ $\qquad$ kan separeras.

$$\int \frac{dy}{y} = \int \frac{-dx}{1+x}$$

$$\ln|y| = -\ln|1+x| + C$$

$$e \qquad e$$

$$|y| = \cancel{e^{\ln}} |1+x|^{-1} \cdot e^{C}$$ $\qquad$ där $C_1 = \pm e^{C}$

$$y = C_1 \cdot |1+x|^{-1} = \frac{C_1}{1+x}$$

b.v. $\quad y(0) = 4 = C_1$

$$\therefore \quad \underline{\underline{y(x) = \frac{4}{1+x}}}$$

3. $$\sum_{n=0}^{\infty} \underbrace{\frac{(n+1)^2}{n!}}_{=a_n} = \frac{1}{0!} + \frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \cdots$$

$a_n = \frac{(n+1)^2}{n!} \to 0$ då $n \to \infty$ ty $n!$ växer snabbare än n^2.

Kvotkriteriet: (bra vid fakulteter)

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+2)^2}{(n+1)!}}{\frac{(n+1)^2}{n!}} = \frac{(n+2)^2 \quad \cancel{n!}}{(n+1)^2 \; (n+1)\cdot \cancel{n!}} = \frac{n^2+4n+4}{n^3+3n^2+3n+1}$$

$(n+1)! = (n+1)\cdot n!$

$$= \frac{\cancel{n^2}(1+\frac{4}{n}+\frac{4}{n^2})}{n^{\cancel{3}}(1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3})} \to 0 < 1$$ då $n \to \infty$

$\therefore$ serien är konv. enl. kvotkriteriet.

3b) $$\lim_{x\to 0} \frac{x\cdot e^{-x} - \sin x}{x^2}$$

$$e^{-x} = 1 - x + \frac{x^2}{2} + O(x^3)$$

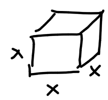
$$\sin x = x - \frac{x^3}{3!} + O(x^5)$$

$$\lim_{x\to 0} \frac{x\cdot(1-x+\frac{x^2}{2}+O(x^3)) - (x-\frac{x^3}{6}+O(x^5))}{x^2}$$

$$\lim_{x\to 0} \frac{\cancel{x} - x^2 + \frac{x^3\cdot 3}{2\cdot 3} + O(x^4) - \cancel{x} + \frac{x^3}{6}}{x^2} =$$

$$= \lim_{x\to 0} \frac{-x^2 + \frac{4x^3}{6} + O(x^4)}{x^2} = \lim_{x\to 0} \frac{\cancel{x^2}(-1+\frac{4}{6}x + O(x^2))}{\cancel{x^2}}$$

$$= \underline{\underline{-1}}$$

6.  iskub.

$V = x^3$

$A = 6 \cdot x^2$

- $\underbrace{\text{Kubens volym avtar med hast.}}_{\frac{dV}{dt}}$ prop. mot begränsningsyta $= -k \cdot A$

$$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt}$$

$$= 3x^2 \cdot \frac{dx}{dt} \qquad = -k \cdot 6x^2$$

$$\therefore\ 3x^2 \frac{dx}{dt} = -k \cdot 6x^2$$

$$\frac{dx}{dt} = -2k$$

$$\underline{x(t) = -2kt + C} \quad (*)$$

b.v. $V(0) = V_0 = x_0^3 = (-2k \cdot 0 + C)^3 \iff \underline{\underline{C = x_0}}$

b.v.2 $V(1) = \frac{3}{4} V_0 = (-2 \cdot k \cdot 1 + x_0)^3$

$$\frac{3}{4} x_0^3 = (x_0 - 2k)^3 \qquad \text{tredjeroten ur....}$$

$$\left(\frac{3}{4}\right)^{1/3} \cdot x_0 = x_0 - 2k \qquad \text{Lös ut } k$$

$$2k = x_0 - \left(\frac{3}{4}\right)^{1/3} \cdot x_0$$

$$k = \frac{1}{2} x_0 \left(1 - \left(\frac{3}{4}\right)^{1/3}\right) \qquad \text{ins i } (*)$$

$$\underline{x(t) = \overset{C}{x_0} - \underbrace{x_0 \left(1 - \left(\frac{3}{4}\right)^{1/3}\right)}_{2k} t}$$

Sök tid T då $x(T) = 0$

$$0 = x_0 - x_0 \left(1 - \left(\frac{3}{4}\right)^{1/3}\right) T \qquad \text{Lös ut } T.$$

$$x_0 \cdot \left(1 - \left(\frac{3}{4}\right)^{1/3}\right) \cdot T = x_0$$

$$T = \frac{\cancel{x_0}}{\cancel{x_0} \cdot \left(1 - \left(\frac{3}{4}\right)^{1/3}\right)} = \frac{1}{1 - \left(\frac{3}{4}\right)^{1/3}} \approx 10{,}93$$

<u>Svar</u>: Iskuben har smalt helt efter ca 10,93 h.