Forts. A4

$$\boxed{\begin{aligned}&\text{Kvadratkomplettering}\\ &x^2+2x+2 = \underbrace{(x+1)^2}_{x^2+2x+1} - 1 + 2\\ &\qquad = (x+1)^2+1\end{aligned}}$$

$$\int\left(\frac{1/2}{x} + \frac{-\frac{1}{2}x-1}{x^2+2x+2}\right)dx$$

$$= \frac{1}{2}\ln|x| + \underbrace{\int\frac{-\frac{1}{2}x-1}{(x+1)^2+1}\,dx}_{=I}$$

$$I = \int\frac{-\frac{1}{2}(x+1)+\frac{1}{2}-1}{(x+1)^2+1}\,dx = \int\frac{-\frac{1}{2}(x+1)-\frac{1}{2}}{(x+1)^2+1}\,dx$$

$$= \left[\begin{aligned}&u = x+1\\ &\frac{du}{dx} = 1\\ &du = dx\end{aligned}\right] = \int\frac{-\frac{1}{2}u-\frac{1}{2}}{u^2+1}\,du$$

$$= \int\left(\frac{-\frac{1}{2}u}{u^2+1} + \frac{-\frac{1}{2}}{u^2+1}\right)du$$

$$= -\frac{1}{2}\ln|u^2+1|\,\frac{1}{2} - \frac{1}{2}\arctan u + C$$

$$= -\frac{1}{4}\ln\left((x+1)^2+1\right) - \frac{1}{2}\arctan(x+1) + C$$

dvs

$$\int\left(\frac{1/2}{x} + \frac{-\frac{1}{2}x-1}{x^2+2x+2}\right)dx = \frac{1}{2}\ln|x| - \frac{1}{4}\ln\left(x^2+2x+2\right) - \frac{1}{2}\arctan(x+1) + C$$

Ex:

$$\int \cos^2 x\,(-\sin x)\,dx = \left[\begin{array}{l} u = \cos x \\ \frac{du}{dx} = -\sin x \\ du = (-\sin x)\,dx \end{array}\right]$$

$$= \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\cos^3 x + C$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\cos 2x = 2\cos^2 x - 1 \Rightarrow \cos^2 x = \frac{\cos 2x + 1}{2}$$

$$\cos 2x = 1 - 2\sin^2 x \Rightarrow \sin^2 x = \frac{1-\cos 2x}{2}$$

$$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx$$

$$= \frac{1}{2}\left(x + (\sin 2x)\,\frac{1}{2}\right) + C = \frac{x}{2} + \frac{\sin 2x}{4} + C$$

Primitiv funktion av typen

$$\int \sin^n x\,\cos^m x\,dx$$

Om n eller m är udda $\Rightarrow$ Substitutionsmetoden

Ex:

$$\int \cos^5 x\,dx = \int \cos x\,(\cos^2 x)^2\,dx$$

$$= \int \cos x\,(1-\sin^2 x)^2\,dx = \left[\begin{array}{l} u = \sin x \\ \frac{du}{dx} = \cos x \\ du = \cos x\,dx \end{array}\right]$$

$$= \int (1-u^2)^2 \, du = \int (1 - 2u^2 + u^4) \, du$$

$$= u - 2u^3 \frac{1}{3} + u^5 \cdot \frac{1}{5} + C$$

$$= \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + C$$

Ex:

$$\int \sin^7 x \cdot \cos^4 x \, dx = \int \sin x \left(\sin^2 x\right)^3 \cos^4 x \, dx$$

$$= \int \sin x \left(1 - \cos^2 x\right)^3 \cos^4 x \, dx = \begin{bmatrix} u = \cos x \\ \frac{du}{dx} = -\sin x \\ -du = \sin x \, dx \end{bmatrix}$$

$$= -\int (1-u^2)^3 u^4 \, du = \ldots$$

$$\int \sin^n x \cos^m x$$

Om både n och m är jämna.

$$\boxed{\begin{aligned} \sin x \cdot \cos x &= \frac{1}{2} \sin 2x \\ \cos^2 x &= \frac{1}{2}(1 + \cos 2x) \\ \sin^2 x &= \frac{1}{2}(1 - \cos 2x) \end{aligned}}$$

Ex:

$$\int \sin^2 x \cos^2 x \, dx = \int (\sin x \cos x)^2 \, dx$$

$$= \int \left(\frac{1}{2}\sin 2x\right)^2 dx = \frac{1}{4}\int \sin^2 2x \, dx$$

$$= \frac{1}{4}\int \frac{1}{2}(1+\cos 4x)\, dx$$

$$= \frac{1}{8}\left(x + \sin 4x \, \frac{1}{4}\right) + C$$

Härledning

(1) $\cos(x+y) = \cos x \cdot \cos y - \sin x \cdot \sin y$

(2) $\cos(x-y) = \cos x \cdot \cos y + \sin x \cdot \sin y$

(1)+(2) $\cos(x+y) + \cos(x-y) = 2\cos x \cdot \cos y$

(2)−(1) $\cos(x-y) - \cos(x+y) = 2 \sin x \cdot \sin y$

(3) $\sin(x+y) = \sin x \cos y + \sin y \cos x$

(4) $\sin(x-y) = \sin x \cos y - \sin y \cos x$

(3)+(4) $\sin(x+y) + \sin(x-y) = 2 \sin x \cos y$

Exempel

$$\int \sin x \cos 2x \, dx = \int \frac{1}{2}\left(\sin(x-2x) + \sin(x+2x)\right) dx$$

$$= \frac{1}{2}\int \left(\underbrace{\sin(-x)}_{-\sin x} + \sin 3x\right) dx$$

$$= \frac{1}{2}\left(\cos x - \cos(3x)\frac{1}{3}\right) + C$$

$$= \frac{1}{2}\cos x - \frac{1}{6}\cos 3x + C$$

Ex:

$$\int \cos^3 2x \sin 2x \, dx = \left[\begin{array}{l} u = \cos 2x \\ \frac{du}{dx} = (-\sin 2x)2 \\ -\frac{1}{2} du = \sin 2x \, dx \end{array}\right]$$

$$= \int u^3\left(-\frac{1}{2}\right) du$$

$$= -\frac{1}{2} u^4 \cdot \frac{1}{4} + C = -\frac{1}{8}\cos^4 2x + C$$

Ex:

$$\dots = \int \frac{1}{1+\tan x}\,dx = \left[\begin{array}{l} u = \tan x \\ \vdots \\ dx = \dfrac{1}{1+u^2}\,du \end{array}\right]$$

$$= \int \frac{1}{1+u}\cdot\frac{1}{1+u^2}\,du$$

$$\frac{1}{(1+u)(1+u^2)} = \frac{A}{u+1} + \frac{Bu+C}{u^2+1}$$

$$= \frac{A(u^2+1)}{(u+1)(u^2+1)} + \frac{(Bu+C)(u+1)}{(u+1)(u^2+1)}$$

$$= \frac{A(u^2+1) + B(u^2+u) + C(u+1)}{(u+1)(u^2+1)}$$

$$= \frac{(A+B)u^2 + (B+C)u + (A+C)}{(u+1)(u^2+1)}$$

$$\begin{array}{l} (u^2) \\ (u) \\ (1) \end{array}\left\{\begin{array}{llll} A+B & & =0 & -1 \\ & B+C & =0 & \\ A & +C & =1 & \end{array}\right. \qquad \left\{\begin{array}{llll} A+B & & =0 & \\ & B+C & =0 & ! \\ & -B+C & =1 & \end{array}\right.$$

$$\left\{\begin{array}{lll} A+B & & =0 \\ & B+C & =0 \\ & 2C & =1 \end{array}\right. \qquad \begin{array}{l} C = \frac{1}{2} \\ B = -\frac{1}{2} \\ A = \frac{1}{2} \end{array}$$

$$= \int \frac{1}{1+u} \cdot \frac{1}{u^2+1}\, du = \int \left(\frac{1/2}{u+1} + \frac{-\frac{1}{2}u + \frac{1}{2}}{u^2+1} \right) du$$

$$= \frac{1}{2} \int \left(\frac{1}{u+1} - \frac{u}{u^2+1} + \frac{1}{u^2+1} \right) du$$

$$= \frac{1}{2} \left(\ln|u+1| - \ln|u^2+1|\, \frac{1}{2} + \arctan u \right) + C$$

$$= \frac{1}{2} \ln|\tan x + 1| - \frac{1}{4} \ln(\tan^2 x + 1) + \frac{1}{2} \arctan(\tan x) + C$$

Klart!