Rep A7

1. Medelvärdet av $f(x) = 2x(1-x)$ på intervallet $[0, 1]$

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{1-0}\int_0^1 2x(1-x)\,dx$$

$$= \int_0^1 (2x - 2x^2)\,dx = \left[2x^2 \cdot \frac{1}{2} - 2x^3 \cdot \frac{1}{3}\right]_0^1$$

$$= 1^2 - \frac{2}{3}1^3 - (0-0) = \frac{1}{3}$$

2. Finn x så att $f(x) = \bar{f}$

$$2x(1-x) = \frac{1}{3}$$

$$2x - 2x^2 = \frac{1}{3}$$

$$x^2 - x + \frac{1}{6} = 0$$

$$x = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{6}} = \frac{1}{2} \pm \sqrt{\frac{3}{12} - \frac{2}{12}}$$

$$= \frac{1}{2} \pm \sqrt{\frac{1}{12}} = \frac{1}{2} \pm \frac{1}{2\sqrt{3}}$$

$$x = \frac{1}{2} + \frac{1}{2\sqrt{3}} \quad \text{eller} \quad x = \frac{1}{2} - \frac{1}{2\sqrt{3}}$$

Medelvärdessatsen $\quad a < c < b$

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c) \quad \Big| \quad \int_a^b f(x)\,dx = (b-a)f(c)$$

Sista steget för Analysens huvudsats

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= -\int_c^a f(x)\,dx + \int_c^b f(x)\,dx$$

Primitiv funktion från areafunktionen

$$= -F(a) + F(b)$$

Ex:

- $\int_0^1 (2x+6x^4+5)\,dx = \left[2\, x^2 \cdot \frac{1}{2} + 6\, x^5 \cdot \frac{1}{5} + 5x\right]_0^1$

$$= \left(1^2 + \frac{6}{5} 1^5 + 5 \cdot 1\right) - (0+0+0) = 7 + \frac{1}{5}$$

- $\int_1^2 \frac{x^2+1}{x^2}\,dx = \int_1^2 (1+x^{-2})\,dx = \left[x + x^{-1}\frac{1}{-1}\right]_1^2$

$$= \left(2 - \frac{1}{2}\right) - \left(1 - \frac{1}{1}\right) = \frac{3}{2}$$

- $\int_0^{\pi/2} \cos x = \left[\sin x\right]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$

- $\int_1^5 \sqrt{x-1}\,dx = \int_1^5 (x-1)^{1/2}\,dx = \left[\frac{1}{3/2}(x-1)^{3/2}\frac{1}{1}\right]_1^5 = \frac{2}{3}4^{3/2} - \frac{2}{3}0^{3/2} = \frac{16}{3}$

Ex:

$$A = 4\int_0^R \sqrt{R^2-x^2}\,dx = \left[\begin{array}{l} x = R\sin\theta \\ \frac{dx}{d\theta} = R\cos\theta \\ dx = R\cos\theta\,d\theta \end{array}\right| \left.\begin{array}{l} x=0 \Rightarrow \theta = 0 \\ x=R \Rightarrow \theta = \frac{\pi}{2} \end{array}\right]$$

$$= 4\int_0^{\pi/2} \sqrt{R^2 - R^2\sin^2\theta}\; R\cos\theta\,d\theta$$

$$= 4\int_0^{\pi/2} \sqrt{R^2\underbrace{(1-\sin^2\theta)}_{\cos^2\theta}}\; R\cos\theta\,d\theta$$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$$= 4\int_0^{\pi/2} R\cos\theta\; R\cos\theta\,d\theta = 4\int_0^{\pi/2} R^2\cos^2\theta\,d\theta$$

$$= 4\int_0^{\pi/2} R^2\frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \frac{4}{2}R^2\left[\theta + \sin 2\theta\;\frac{1}{2}\right]_0^{\pi/2}$$

$$= 2R^2\left(\frac{\pi}{2} + \frac{1}{2}\underbrace{\sin\pi}_{=0} - \left(0 + \frac{1}{2}\sin 0\right)\right)$$

$$= 2R^2\cdot\frac{\pi}{2} = \pi\cdot R^2$$

Ex:

$$\int_0^1 \frac{\sqrt{x}}{(\sqrt{x}+1)^4}\,dx = \left[\begin{array}{l|l} u=\sqrt{x}+1 & x=0 \\ x=(u-1)^2 & \Downarrow\ u=1 \\ \frac{dx}{du}=2(u-1) & x=1 \\ dx=2(u-1)\,du & \Downarrow\ u=2 \end{array}\right]$$

$u=\sqrt{x}+1$
$u-1=\sqrt{x}$
$(u-1)^2=x$

$$= \int_1^2 \frac{u-1}{u^4}\,2(u-1)\,du = 2\int_1^2 \frac{1}{u^4}(u^2-2u+1)\,du$$

$$= 2\int_1^2 (u^{-2}-2u^{-3}+u^{-4})\,du = 2\left[\frac{1}{-1}u^{-1} - 2\frac{1}{-2}u^{-2} + \frac{1}{-3}u^{-3}\right]_1^2$$

$$= 2\left(\left(-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{3}\frac{1}{2^3}\right)-\left(-1+2-\frac{1}{3}\right)\right)$$

$$= 2\left(-\frac{1}{2}+\frac{1}{4}-\frac{1}{24}+1-2+\frac{1}{3}\right)$$

$$= 2\left(-\frac{12}{24}+\frac{6}{24}-\frac{1}{24}+1-2+\frac{8}{24}\right) = 2\left(-1+\frac{1}{24}\right)$$

$$= 2\left(-\frac{23}{24}\right) = -\frac{23}{12}$$

Ex:

$$\int_0^{1/2} \cos(\pi \cdot x)\, e^{2\sin(\pi \cdot x)}\, dx = \left[\begin{array}{l} u = \sin(\pi \cdot x) \\ \frac{du}{dx} = \cos(\pi \cdot x) \cdot \pi \\ \frac{1}{\pi} \cdot du = \cos(\pi \cdot x) \cdot dx \end{array} \middle| \begin{array}{l} x=0 \\ u=0 \\ x=\frac{1}{2} \\ u=1 \end{array} \right]$$

$$= \int_0^1 \frac{1}{\pi} e^{2u}\, du = \frac{1}{\pi}\left[e^{2u} \frac{1}{2} \right]_0^1 = \frac{1}{\pi}\left(\frac{1}{2} e^{2\cdot 1} - \frac{1}{2} \underbrace{e^{2\cdot 0}}_{1} \right)$$

$$= \frac{1}{\pi}\left(\frac{1}{2} e^2 - \frac{1}{2} \right) = \frac{1}{2\pi}\left(e^2 - 1 \right)$$