C0002M – Numerical analysis, Lecture 11

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Numerical integration (quadrature)

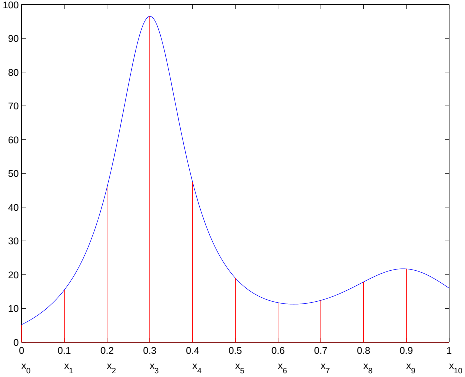
When the definite integral $\int_a^b f(x)\,dx$, is approximated, we make use of an evenly spaced partitioning of the interval $[a, b]$, consisting of n subintervals, each with width $h = (b-a)/n$, so

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

and

$$x_{k+1} - x_k = h$$

for all k.



Midpoint rule

Evaluate the function value at the middle of each subinterval:

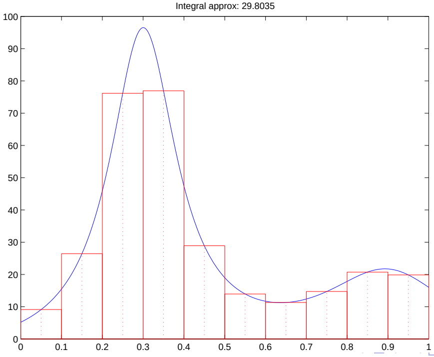
$$x_k^* = (x_{k-1} + x_k)/2, \qquad y_k^* = f(x_k^*).$$

Approximation on one subinterval

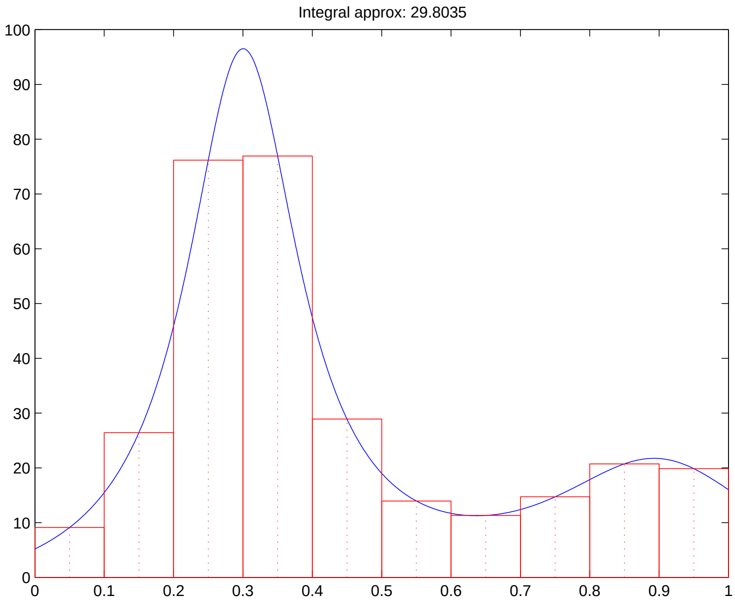
$$\int_{x_{k-1}}^{x_k} f(x)\,dx \approx h \cdot y_k^*.$$

Sum up to get the composite rule

$$\int_a^b f(x)\,dx \approx h\,(y_1^* + y_2^* + \cdots + y_n^*).$$



Midpoint rule



Trapezoidal rule

Evaluate the function values at the partitioning points:

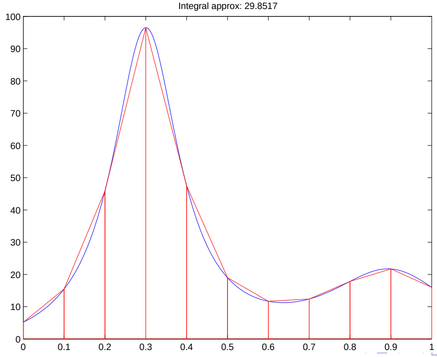
$$y_k = f(x_k).$$

Approximation on one subinterval

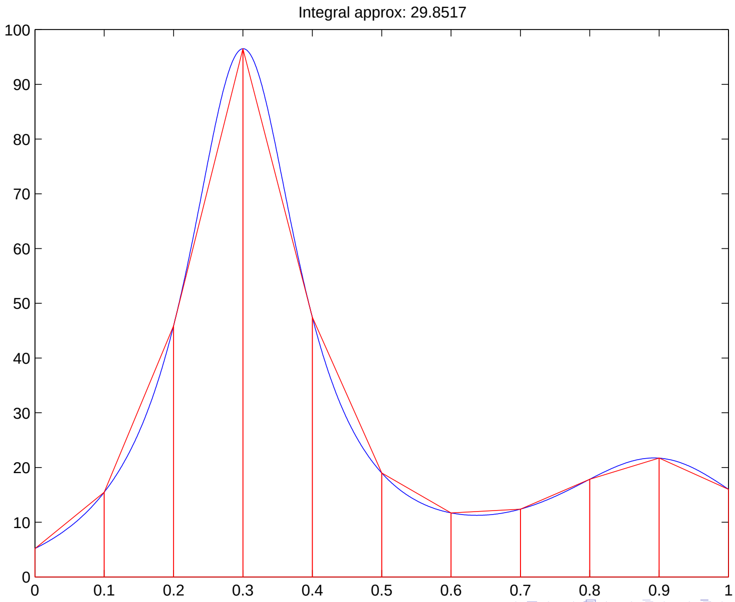
$$\int_{x_{k-1}}^{x_k} f(x)\,dx \approx h/2\,(y_{k-1} + y_k).$$

Sum up to get the composite rule

$$\int_a^b f(x)\,dx \approx h/2\,(y_0 + 2\,y_1 + 2\,y_2 + \cdots + 2\,y_{n-1} + y_n).$$



Trapezoidal rule



Simpson's rule

Let: $y_k = f(x_k)$.

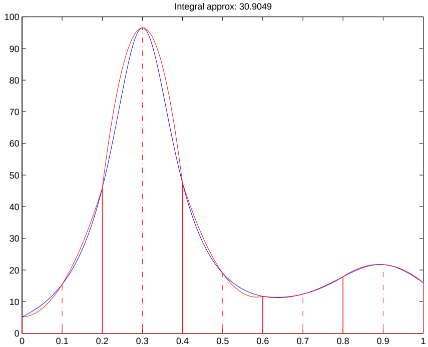
Approximation over two subintervals

$$\int_{x_{k-1}}^{x_{k+1}} f(x)\,dx \approx h/3\,(y_{k-1} + 4\,y_k + y_{k+1}).$$

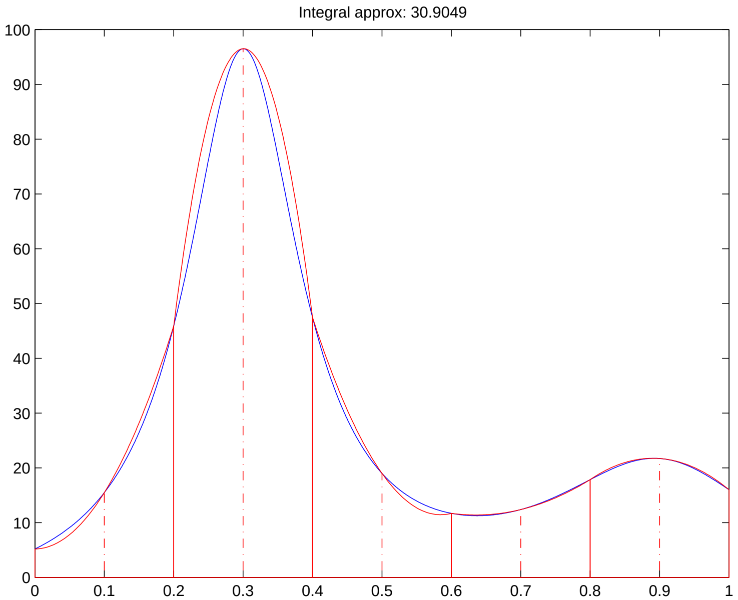
This is the integral of the interpolating polynomial of degree 2.

Sum up to get the composite rule

$$\int_a^b f(x)\,dx \approx h/3\,(y_0 + 4\,y_1 + 2\,y_2 + 4\,y_3 + 2\,y_4 + \cdots + 4\,y_{n-1} + y_n).$$



Simpson's rule



Simpson's 3/8 rule

Let: $y_k = f(x_k)$.

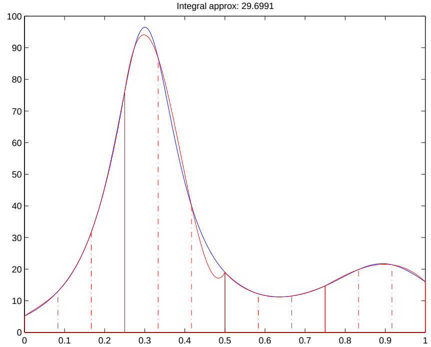
Approximation over three subintervals

$$\int_{x_0}^{x_3} f(x)\,dx \approx 3\,h/8\,(y_0 + 3\,y_1 + 3\,y_2 + y_3).$$

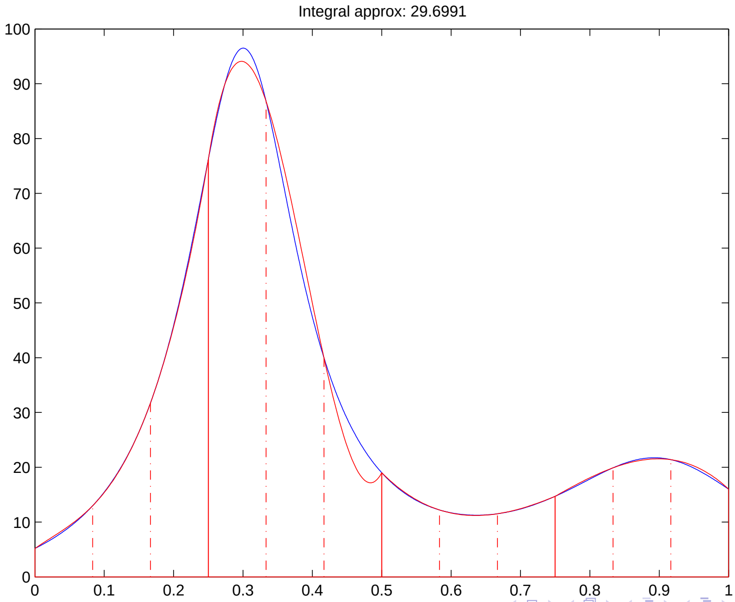
This is the integral of the interpolating polynomial of degree 3.

Sum up to get the composite rule

$$\int_a^b f(x)\,dx \approx 3\,h/8\,(y_0 + 3\,y_1 + 3\,y_2 + 2\,y_3 + 3\,y_4 + \cdots + 3\,y_{n-1} + y_n).$$



Simpson's 3/8 rule



Error estimation

The errors for the composite version of these methods can be expressed as below. The value of ξ is unknown, apart from that it is in the open interval (a, b), and varies little as h becomes smaller

Midpoint rule: $$\frac{b-a}{24} f''(\xi)\, h^2 = O(h^2)$$

Trapezoidal rule: $$-\frac{b-a}{12} f''(\xi)\, h^2 = O(h^2)$$

Simpson's rule: $$-\frac{b-a}{180} f^{(4)}(\xi)\, h^4 = O(h^4)$$

Simpson's 3/8 rule: $$-\frac{b-a}{80} f^{(4)}(\xi)\, h^4 = O(h^4)$$

Newton-Cotes formulas

The Midpoint rule, Trapezoidal rule and the Simpson rules are all examples of Newton-Cotes formulas, where the function is evaluated at equidistant points, and the integral is approximated by a interpolation polynomial that passes through the function values at those points.

Closed methods

Closed methods evaluate the function at the endpoints of the interval. Ex. the Trapezoidal rule and the Simpson rules. See Table 17.2 in Chapra.

Open methods

Open methods exclude the endpoints of the interval. Ex. the Midpoint rule. See Table 17.4 in Chapra. They are not as common as closed methods.

Estimate integrals over unequal segments

Either

Since h_i varies, use a single trapezoidal rule (not composite) in each interval and sum up.

or

Find an interpolating function and integrate that one.

For piecewise interpolation (linear, cubic spline), it is simple enough to find the integral of each interval analytically.