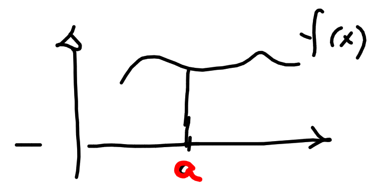
$\boxed{F12}$ Partiell integration (FN kap 5.2)

Elementära räkneregler (kap 6.2)

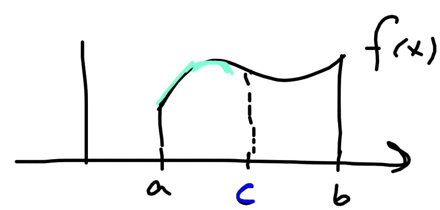
- $\displaystyle\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

- $\displaystyle\int_a^b c\cdot f(x)\,dx = c\cdot\int_a^b f(x)\,dx$ $\quad c =$ konstant.

- $\displaystyle\int_a^a f(x)\,dx = 0$

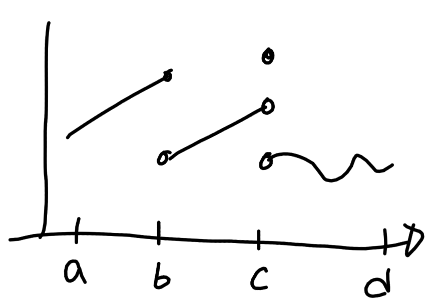


- $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$



- $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

integralen byter tecken om integrationsgränserna byter plats.



- f styckvis kontinuerlig fkn kan integreras, uppdelad i intervall.

ex) $\int \frac{dx}{2x-3} = \frac{\ln|2x-3|}{2} + C$

$\boxed{\int \frac{1}{x} = \ln|x| + C}$

1:a grads uttryck

$\boxed{\int \frac{dx}{1+x^2} = \arctan(x) + C}$

kvadrat + 1 i nämnaren.

ex) $\int \frac{dx}{16+x^2} = \frac{1}{16} \cdot \int \frac{dx}{1+\frac{x^2}{16}} = \frac{1}{16} \cdot \int \frac{dx}{\underbrace{1+\left(\frac{x}{4}\right)^2}_{\text{kvadrat + 1 i nämnaren}}} = \frac{1}{16} \frac{\arctan\left(\frac{x}{4}\right)}{\frac{1}{4}} + C$

ex) $\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x+1)^2+1} = \arctan(x+1) + C$

kvadratkomplettera

Partiell integration

Används då $\underbrace{\text{integranden}}$ är en produkt av två "olika" funktioner. $\int f(x)\,dx$
(inte inre derivata till varandra).

ex) $\int x \cdot \cos x \, dx$

Härledning:

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = \underbrace{f'(x)\cdot g(x) + f(x)\cdot g'(x)}$$

$$C + f(x)\cdot g(x) = \int f'(x)\cdot g(x)\,dx + \int f(x)\,g'(x)\,dx$$

$$\int f'(x)\cdot g(x)\,dx = f(x)\cdot g(x) - \int f(x)\cdot g'(x)\,dx + C$$

integ. derivera

Partiell integration:

int. der

$$\boxed{\int f(x)\cdot g(x)\,dx = F(x)\cdot g(x) - \underbrace{\int F(x)\cdot g'(x)\,dx}}$$

börja alltid med funktionen som ska integreras.

kvarvarande integral (enklare).

der int

ex) $\int x \cdot \cos x \, dx = \sin x \cdot x - \int \sin x \cdot 1 \, dx$

$= \sin x \cdot x - (-\cos x) + C$

$= \sin x \cdot x + \cos x + C$

koll: $\frac{d}{dx}\left(\sin x \cdot x + \cos x + C\right) =$

$= \cos x \cdot x + \sin x \cdot 1 - \sin x + 0 = x \cdot \cos x$ OK.

ex) $\int \overset{\text{int.}}{x^2} \cdot \overset{\text{der}}{\ln x}\, dx = \frac{x^3}{3} \cdot \ln x - \underbrace{\int \frac{x^3}{3} \cdot \frac{1}{x}\, dx}$

$= \frac{x^3}{3} \cdot \ln x - \frac{1}{3} \int x^2\, dx$

$= \frac{x^3}{3} \cdot \ln x - \frac{1}{3} \cdot \frac{x^3}{3} + C$

ex) $\int \overset{\text{int}}{1} \cdot \overset{\text{der}}{\ln x}\, dx = x \cdot \ln x - \int x \cdot \frac{1}{x}\, dx$

$= x \cdot \ln x - \int 1 \cdot dx$

$= x \cdot \ln x - x + C$

ex) $\int \overset{\text{int}}{1} \cdot \overset{\text{der}}{\arctan x}\, dx = x \cdot \arctan x - \underbrace{\int x \cdot \frac{1}{1+x^2}\, dx}$

enkel att lösa med variabelbyte el. sid 248

ex) $I = \int \overset{\text{int}}{e^x} \cdot \overset{\text{der}}{\cos x}\, dx \overset{\text{P.I}}{=} e^x \cdot \cos x - \int \overset{\text{int}}{e^x} \cdot \overset{\text{der}}{(-\sin x)}\, dx \overset{\text{P.I}}{=}$

$= e^x \cdot \cos x + \overbrace{e^x \cdot \sin x - \underbrace{\int e^x \cdot \cos x\, dx}_{= I}}$

$I = e^x \cdot \cos x + e^x \cdot \sin x - I$

$2I = \quad$ – " –

$I = \frac{1}{2}\left(e^x \cos x + e^x \cdot \sin x\right) + C$

$$\int_0^{2\pi} \sin x \, dx = 0$$

der int

ex) $$\int x \cdot \sin\left(\frac{x}{2}\right) dx = -x \cdot \frac{\cos\left(\frac{x}{2}\right)}{\frac{1}{2}} - \int \frac{-\cos\left(\frac{x}{2}\right)}{\frac{1}{2}} \cdot 1 \, dx$$

$$= -2x \cdot \cos\left(\frac{x}{2}\right) + 2 \cdot \int \cos\left(\frac{x}{2}\right) dx$$

$$= -2x \cdot \cos\left(\frac{x}{2}\right) + 2 \cdot \frac{\sin\left(\frac{x}{2}\right)}{\frac{1}{2}} + C$$

$$= \underbrace{-2x \cdot \cos\left(\frac{x}{2}\right) + 4 \sin\left(\frac{x}{2}\right) + C}.$$

Koll $$\frac{d}{dx}(HL) = -2 \cdot \cos\frac{x}{2} + (-2x)\cdot\left(-\sin\frac{x}{2}\right)\cdot\frac{1}{2} + 4 \cdot \cos\left(\frac{x}{2}\right)\cdot\frac{1}{2} + 0$$

$$= -2 \cancel{\cos\frac{x}{2}} + \underbrace{x \sin\frac{x}{2}} + 2 \cancel{\cos\frac{x}{2}} = \text{integranden}$$

OK

int der

ex) $$I = \int x^3 \cdot (\ln x)^2 \, dx \overset{P.I.}{=} \underbrace{\frac{x^4}{4} \cdot (\ln x)^2} - \underbrace{\int \frac{x^4}{4} \cdot 2 \cdot (\ln x) \cdot \frac{1}{x} \, dx}_{I_2}$$

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int der

$$I_2 = \frac{1}{2} \int x^3 \cdot \ln x \, dx \overset{P.I}{=} \frac{1}{2}\left[\frac{x^4}{4} \cdot \ln x - \underbrace{\int \frac{x^4}{4} \cdot \frac{1}{x} \, dx}\right]$$

$$-\frac{1}{4} \int x^3 \, dx$$

$$-\frac{1}{4} \cdot \frac{x^4}{4} + C$$

$$I = \underbrace{\frac{x^4}{4} (\ln x)^2} - \frac{1}{2}\left[\frac{x^4}{4} \cdot \ln x - \frac{1}{16} x^4\right] + C$$

ex) $f(x)$ har minvärdet 2.

och $f'(x) = x^2 - 2x - 3$

Sök: $f(x)$.

$f'(x) = 0$ då $f(x)$ har min/max

$x^2 - 2x - 3 = 0$

$\begin{cases} x_1 = 3 \\ x_2 = -1 \end{cases}$

x		-1		3	
$f'(x) = (x-3)(x+1)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	lok max	$\searrow$	lok min	$\nearrow$

min el max ? (Teckenstud av derivatan, alt. 2:a derivatan)

$f''(x) = 2x - 2$

$f''(3) = 2 \cdot 3 - 2 = 4 > 0$ $\cup$ f lok min då $x = 3$

$f''(-1) = 2(-1) - 2 = -4 < 0$ $\cap$ lok max då $x = -1$.

$$f(x) = \int (x^2 - 2x - 3)\,dx = \underline{\frac{x^3}{3} - x^2 - 3x + C}$$

min 2 då $x = 3$

$$f(3) = 2 = \frac{3^3}{3} - 3^2 - 3 \cdot 3 + C$$

$$= \cancel{9} - \cancel{9} - 9 + C = 2$$

$$C = 11$$

$$\therefore f(x) = \frac{x^3}{3} - x^2 - 3x + 11$$

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ex) $f(x) = (x+3)^2$ $F(0) = 10$

$F(x)$ primitiv fkn till $f(x)$

Sök $F(x)$

$$F(x) = \int (x+3)^2\,dx$$

alt.
$= \frac{(x+3)^3}{3} + C_1$

$$= \int (x^2 + 6x + 9)\,dx = \underline{\frac{x^3}{3} + 3x^2 + 9x + C}$$

$$F(0) = 10 = 0^3 + 3 \cdot 0^2 + 9 \cdot 0 + C$$

$$C = 10$$

$$\therefore F(x) = \underline{\frac{x^3}{3} + 3x^2 + 9x + 10}.$$

alt.

$F(0) = 10 =$

$= \frac{(0+3)^3}{3} + C_1$

$= 9 + C_1$

$$F(x) = \frac{(x+3)^3}{3} + 1$$