$\boxed{F13}$ Variabelbyte (substitution)

Rep. Partiell integration (P.I)

där f, g är två olika funktioner

$$\boxed{\int \overset{\text{int}}{f(x)} \cdot \overset{\text{der}}{g(x)}\,dx = F(x)\cdot g(x) - \int F(x)\cdot g'(x)\,dx}$$

Även: ex) $\int (\arctan x)\,dx = \int \overset{\text{int}}{1} \cdot \overset{\text{der}}{\arctan x}\,dx = x \cdot \arctan x - \underbrace{\int x \cdot \frac{1}{1+x^2}\,dx}_{I_2}$

(löses på nästa sida)

...

ex) $\underbrace{\int \frac{\ln x}{x}\,dx}_{I} = \int \overset{\text{int}}{\frac{1}{x}} \cdot \overset{\text{der}}{\ln x}\,dx = \ln x \cdot \ln x - \underbrace{\int \ln x \cdot \frac{1}{x}\,dx}_{I}$

$$I = \ln x \cdot \ln x + C - I$$

$$2I = (\ln x)^2 + C$$

$$I = \frac{1}{2}(\ln x)^2 + C_1 \qquad C_1 = \frac{C}{2}$$

Sammansatta funktioner

inre derivata

$$\frac{d}{dx}\left(\ln(1+x^2)\right) = \frac{1}{(1+x^2)}\cdot(0+2x) = \frac{2x}{1+x^2}$$

integrera båda led.

$$\ln(1+x^2) + C = \int \frac{2x}{1+x^2}\,dx$$

$$\frac{d}{dx} f(g(x)) = f'(g(x))\cdot g'(x)$$

yttre derivata inre derivata

$$\int f(t)\,dt = F(t) + C = F(g(x)) + C$$

$$\boxed{\int \underbrace{f(g(x))}_{\text{sammansatt funktion}}\cdot \underbrace{g'(x)}_{\text{inre derivata till}}\,dx = F(g(x)) + C}$$

Sätt: $\boxed{t = g(x)}$

$\frac{dt}{dx} = g'(x)$

$dt = g'(x)\,dx$

Svårt att se ibland. Lösningen är variabelbyte

Gör variabelbyte i den sammansatta funktionen.

ex) $\int \underline{x}\cdot e^{(-x^2)}\,\underline{dx} = \left\{\begin{array}{l} \text{variabelbyte:} \\ \boxed{t = -x^2} \\ \frac{dt}{dx} = -2x \\ dt = -2\underline{x\cdot dx} \\ \frac{dt}{-2} = \underline{x\cdot dx} \end{array}\right\}$

$= \int e^t \cdot \frac{dt}{-2} =$

$= -\frac{1}{2}e^t + C$

$= \left\{\text{byt tillbaka efter integreringen } \atop \text{primitiv funktion}\right\} = \underline{-\frac{1}{2}e^{-x^2} + C}$

(forts. från sid 1)

ex) $\int (\arctan x)\,dx = \int 1\cdot \arctan x\,dx = x\cdot \arctan x - \underbrace{\int x \cdot \frac{1}{1+x^2}\,dx}_{I_2}$

$$I_2 = \int \frac{x\,dx}{\underbrace{1+x^2}_{t}} \quad (x\,dx = dt/2) \quad = \left\{ \begin{array}{l} \boxed{t = 1+x^2} \\ \frac{dt}{dx} = 0+2x \\ \frac{dt}{2} = \frac{\cancel{2}x\,dx}{\cancel{2}} \end{array} \right\}$$

$$= \int \frac{dt}{2\cdot t} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| + C = \left\{ \begin{array}{l} \text{byt} \\ \text{tillbaka} \end{array} \right\}$$

$$= \frac{1}{2}\ln|1+x^2| + C$$

$\therefore \int \arctan x = x\cdot \arctan x - \frac{1}{2}\ln|1+x^2| + C$

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ex) $\int \tan x\,dx = \int \frac{\sin x\,dx}{\underbrace{\cos x}_{t}} \quad (\sin x\,dx = -dt) \quad = \left\{ \begin{array}{l} \boxed{t = \cos x} \\ \frac{dt}{dx} = -\sin x \\ -dt = \sin x \cdot dx \end{array} \right\}$

$$= \int -\frac{1}{t}\,dt = -\ln|t| + C$$

$\left\{ \text{byt tillbaka} \right\} = -\ln|\cos x| + C$

$$\int e^{\sqrt{x}}\,dx = \left\{ \begin{array}{ll} \boxed{y = \sqrt{x}} = x^{1/2} \Rightarrow & x = y^2 \\ \frac{dy}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\underbrace{\sqrt{x}}_{=y}} & \frac{dx}{dy} = 2y \\ 2dy\cdot y = dx & dx = 2y\,dy \end{array} \right\}$$

inverterbart variabelbyte

$$= \int e^{y}\cdot 2y\,dy \overset{P.I.}{=} e^{y}\cdot 2y - \int e^{y}\cdot 2\,dy$$

(int, der)

$$= e^{y}\cdot 2y - 2\cdot e^{y} + C = \left\{ \begin{array}{l} \text{byt} \\ \text{tillbaka} \end{array} \right\}$$

$$= e^{\sqrt{x}}\cdot 2\sqrt{x} - 2\cdot e^{\sqrt{x}} + C$$

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$$\int \frac{\cos x \cdot \sin x}{(\underbrace{4+\cos^2 x}_{t})^2}\,dx = \left\{ \begin{array}{l} \text{variabelbyte i den} \\ \text{sammansatta funktionen} \\ \boxed{t = 4+\cos^2 x} \\ \frac{dt}{dx} = 2\cdot\cos x\cdot(-\sin x) \\ \frac{dt}{-2} = \cos x \cdot \sin x \cdot dx \end{array} \right.$$

$$= \int \frac{1}{t^2}\frac{dt}{(-2)} =$$

std. integral.

$$= -\frac{1}{2}\int t^{-2}\,dt = -\frac{1}{2}\,\frac{t^{-2+1}}{-2+1} + C = \text{byt tillbaka}$$

$$= +\frac{1}{2}\,t^{-1} + C = \frac{1}{2}(4+\cos^2 x)^{-1} + C = \frac{1}{2(4+\cos^2 x)} + C$$

ex) $$\int x\cdot\sqrt{x-4}\,dx = \left\{ \begin{array}{l} \text{prova byta ut hela rotuttrycket.} \\ \boxed{t=\sqrt{x-4}} \Rightarrow t^2 = x-4 \\ x = t^2+4 \\ \frac{dx}{dt} = 2t \\ dx = 2t\,dt \end{array} \right\}$$

$$= \int (t^2+4)\cdot t\cdot 2t\,dt = \int 2t^2(t^2+4)\,dt = \int (2t^4+8t^2)\,dt$$

$$= 2\cdot\frac{t^5}{5} + 8\cdot\frac{t^3}{3} + C = \left\{ \begin{array}{l}\text{byt} \\ \text{tillbaka}\end{array} \right\} = \frac{2}{5}(x-4)^{5/2} + \frac{8}{3}(x-4)^{3/2} + C$$

Bestämd integral

$$\int_{x=0}^{x=1} \cos(1+\sqrt{x})\,dx = \left\{ \begin{array}{l} \boxed{t = 1+\sqrt{x}} \Leftrightarrow \\ \text{byt gränser:} \\ x=0 \Rightarrow t=1 \\ x=1 \Rightarrow t=2 \end{array} \right.$$

$\sqrt{x} = t-1 \Rightarrow$

$x = (t-1)^2$

$x = t^2-2t+1$

$\frac{dx}{dt} = 2t-2$

$$\boxed{\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b)-F(a)}$$

$$= \int_{t=1}^{t=2} \underbrace{\cos(t)}_{\text{int.}} \cdot \underbrace{(2t-2)}_{\text{der}}\,dt \overset{\text{P.I}}{=} \left[\sin t \cdot (2t-2)\right]_{t=1}^{t=2} - \int_1^2 \sin t \cdot 2\,dt$$

$$= \left[\sin t\,(2t-2) - 2\cdot(-\cos t) + C\right]_{t=1}^{t=2}$$

$$= \sin 2\cdot(4-2) + 2\cdot\cos 2 + C - \left(\cancel{\sin 1\cdot(2-2)} + 2\cdot\cos 1 + C\right)$$

$$= \underline{2\cdot\sin 2 + 2\cdot\cos 2 + \cancel{C} - 2\cos 1 - \cancel{C}}$$

- $\boxed{\int f(\underbrace{g(x)}_{t}) \cdot \overset{\text{i.d}}{g'(x)}\,dx} = \left\{ \begin{array}{l} \boxed{t = g(x)} \\ \frac{dt}{dx} = g'(x) \\ dt = g'(x)\,dx \end{array} \right\} =$

$= \int f(t) \cdot dt = F(t) + C = \{\text{byt tillbaka}\} = \boxed{F(g(x)) + C}$

- $\boxed{\int \frac{\overset{\text{i.d}}{u'(x)}}{u(x)}\,dx} = \left\{ \begin{array}{l} t = u(x) \\ dt = u'(x)\,dx \end{array} \right\} = \int \frac{1}{t}\,dt = \ln|t| + C$

$= \ln|u(x)| + C$

ex) $\int \frac{1}{2x+4}\,dx = \frac{\ln|2x+4|}{2} + C$

ex) $\int \frac{\overset{\text{i.d.}}{2x+7}}{(x^2+7x)}\,dx = \left\{ \begin{array}{l} t = x^2+7x \\ \frac{dt}{dx} = 2x+7 \end{array} \right\} = \int \frac{dt}{t} = \ln|t| + C$

$= \ln|x^2+7x| + C$

- $\boxed{\int [\underbrace{g(x)}_{t}]^p \cdot \overset{\text{i.d.}}{g'(x)}\,dx = \frac{[g(x)]^{p+1}}{p+1} + C}$ $\quad p \neq -1.$