$\boxed{\text{F14}}$ Partialbråksuppdelning Rationella integrander

alt: $t = e^x$

Rep. $\int e^{2x}\sqrt{1+e^x}\,dx = \left\{ \begin{array}{l} \boxed{t = 1+e^x} \\ \frac{dt}{dx} = e^x \\ dt = e^x \cdot dx \end{array} \right. \iff \left. \begin{array}{l} e^x = t-1 \\ \text{alt: } x = \ln|t-1| \\ \frac{dx}{dt} = \frac{1}{t-1} \\ dx = \frac{dt}{t-1} \end{array} \right\}$

$e^{2x} = (e^x)^2$

$$= \int e^x \cdot e^x \cdot \sqrt{1+e^x} \cdot dx =$$

$$= \int (t-1)\cdot\sqrt{t}\,dt$$

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$$\int (t-1)^{\cancel{2}}\sqrt{t}\,\frac{dt}{\cancel{t-1}} = \int (t-1)\sqrt{t}\,dt = \int \left(t\cdot t^{1/2} - t^{1/2}\right) dt$$

$$\boxed{\begin{array}{l} t^a \cdot t^b = t^{a+b} \\ 10^2 \cdot 10^3 = 10^{2+3} \end{array}}$$

$$= \int \left(t^{3/2} - t^{1/2}\right) dt = \frac{t^{5/2}}{5/2} - \frac{t^{3/2}}{3/2} + C$$

$$= \left\{ \text{byt tillbaka} \right\} = \underline{\frac{2}{5}\left(1+e^x\right)^{5/2} - \frac{2}{3}\left(1+e^x\right)^{3/2} + C}$$

ex) $\int \sin^3 x \, dx$

Sinus eller cosinus med:

- Udda exponent – skriv om m.h.a. trig. ettan.

jfr. $\int \sin^2 x \, dx$

- Jämn exponent – skriv om m.h.a dubbla vinkeln

forts. $\int \sin^3 x \, dx = \left\{ \begin{array}{l} \cos^2 x + \sin^2 x = 1 \\ \sin^2 x = 1 - \cos^2 x \end{array} \right\}$

$= \int \sin x \cdot (1 - \cos^2 x) \, dx = \underbrace{\int \sin x \, dx}_{I_1} - \underbrace{\int \sin x \cdot \cos^2 x \, dx}_{I_2}$

$I_1 = -\cos x + C_1$

$I_2 = \int \sin x \cdot \cos^2 x \, dx = \left\{ \begin{array}{l} \boxed{t = \cos x} \\ \frac{dt}{dx} = -\sin x \\ -dt = \sin x \, dx \end{array} \right\}$

$= -\int t^2 \, dt = -\frac{t^3}{3} + C_2 = -\frac{\cos^3 x}{3} + C_2$

$I = I_1 - I_2 = -\cos x - \left(-\frac{\cos^3 x}{3}\right) + C_3$

$$\boxed{\cdot \int \frac{f'(x)}{f(x)}\,dx} = \left\{ \begin{array}{l} t = f(x) \\ dt = f'(x) \cdot dx \end{array} \right\} = \int \frac{dt}{t} = \ln|t| + C$$

$$= \boxed{\ln|f(x)| + C}$$

$$\cdot \boxed{\int [f(x)]^p \cdot f'(x)\,dx} = \left\{ \begin{array}{l} t = f(x) \\ dt = f'(x)\,dx \end{array} \right\} = \int t^p\,dt = \frac{t^{p+1}}{p+1} + C$$

$$= \boxed{\frac{[f(x)]^{p+1}}{p+1} + C} \qquad p \neq -1$$

Rationella funktion – kvot av polynom

- $\int \frac{1}{2x+3}\,dx = \frac{\ln|2x+3|}{2} + C$

- $\int \frac{1}{x^2+2x+2}\,dx$ < kvadratkomplettera : $(x+1)^2-1^2+2$; faktorisera nämnaren

$$x^2+ax+b = \left(x+\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 + b$$

$$\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{(x+1)^2+1}\,dx = \arctan(x+1) + C$$

ex) $\int \frac{1}{x^2-6x+11}\,dx = \Big\{$ kvadratkomp. $(x-3)^2-3^2+11 = (x-3)^2+2$

$$\int \frac{1}{(x-3)^2+2}\,dx = \frac{1}{2}\cdot\int \frac{1}{\frac{(x-3)^2}{2}+1}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\left(\frac{x-3}{\sqrt{2}}\right)^2+1}\,dx = \Big\{ \text{kvadrat} + 1 \text{ i nämnaren} \Rightarrow \arctan\ldots$$

$$= \frac{1}{2}\,\underbrace{\arctan\left(\frac{x-3}{\sqrt{2}}\right)}_{\frac{1}{\sqrt{2}}} + C = \frac{\sqrt{2}}{2}\arctan\left(\frac{x-3}{\sqrt{2}}\right) + C$$

Alt: $\boxed{\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\cdot\arctan\left(\frac{x}{a}\right) + C}$

ej minus.

$$\boxed{\int \frac{1}{1+x^2}\,dx = \arctan x + C}$$

ex) $\int \frac{dx}{x^2-2x-3} =$

- kvadratkompl.

$(x-1)^2-1^2-3$

$(x-1)^2-4$ Nej!

ej arctan pga minustecken

- faktorisera:

$x^2-2x-3=0$

$x=1\pm\sqrt{1+3}$

$x=1\pm 2 \begin{cases} 3 \\ -1 \end{cases}$ Ja!

$= \int \frac{1}{(x-3)(x+1)} dx$

Ansats till partialbråk:

ansats.

$$\frac{1}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

liknämnigt som i VL.

$$\frac{1}{(x-3)(x+1)} = \frac{A(x+1)}{(x-3)(x+1)} + \frac{B(x-3)}{(x+1)(x-3)}$$

$$\frac{1}{(x-3)(x+1)} = \frac{A(x+1)+B(x-3)}{(x-3)(x+1)}$$

Nämnarna lika & Täljarna lika.

Identifiera koefficienterna i täljarna.

VL HL

Täljare: $1 = \underline{Ax} + A + \underline{Bx} - 3B$

$$\begin{aligned} x: \quad & 0 = \underline{A} + \underline{B} \\ \text{konst:} \quad & 1 = A - 3B \end{aligned}$$

$$\begin{array}{cc} A & B \end{array}$$
$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -3 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -4 & 1 \end{array}\right)$$

$A=-B$

$A=\frac{1}{4}$

$-4B=1$

$B=-\frac{1}{4}$

$$\therefore \int \frac{dx}{(x-3)(x+1)} = \int \frac{1/4}{x-3} - \frac{1/4}{x+1}\, dx =$$

$$= \frac{1}{4}\cdot \ln|x-3| - \frac{1}{4}\ln|x+1| + C$$

$$= \frac{1}{4} \ln\left|\frac{x-3}{x+1}\right| + C$$

$\ln A - \ln B =$

$= \ln \frac{A}{B}$

Ex) $I = \int \frac{x^3+4}{x^2+x}\,dx =$

Om grad T $\geq$ grad N
gör polynomdivision

Täljaren måste alltså
ha lägre grad än nämnaren.

$$\begin{array}{r|l} & x - 1 \quad \text{kvot} \\ x^2+x & x^3 \qquad + 4 \\ & -(x^3+x^2) \\ & -x^2 \quad + 4 \\ & -(-x^2 - x) \\ & x+4 \quad \text{rest} \end{array}$$

$$I = \int x - 1 + \frac{x+4}{\underbrace{x^2+x}} \,dx = \frac{x^2}{2} - x + \underbrace{\int \frac{x+4}{x(x+1)}\,dx}_{I_2}$$

kvadrat. komp. $(x+\frac{1}{2})^2 - \frac{1}{4}$ Nej. Faktorisering

partialbråks ansats till I_2

$$\frac{x+4}{x\cdot(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$\frac{x+4}{x(x+1)} = \frac{A(x+1) + Bx}{x(x+1)}$$

identifiera koeff. i täljarna.

	VL		HL
x:	1	=	$A + B$
konst:	4	=	A

$A = 4$
$B = 1 - A = -3$

$$I_2 = \int \frac{4}{x} - \frac{3}{x+1}\,dx = 4\cdot\ln|x| - 3\cdot\ln|x+1| + C$$

$I = \ldots\ldots$

$$\int \frac{1}{(x^2-1)(x+1)}\,dx = \int \frac{dx}{(x+1)(x-1)(x+1)} =$$

OBS! Ansats vid dubbelrot

Ansats. PB

$$\frac{1}{(x+1)^2\cdot(x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$$

samma nämnare som i VL

$$\frac{1}{(x+1)^2(x-1)} = \frac{A\overbrace{(x+1)(x-1)}^{(x^2-1)} + B(x-1) + C\overbrace{(x+1)^2}^{x^2+2x+1}}{(x+1)^2(x-1)}$$

$$\frac{1}{(x+1)^2(x-1)} = \frac{A(x^2-1) + B(x-1) + C(x^2+2x+1)}{(x+1)^2(x-1)}$$

Identifiera tälj.

$$\left.\begin{array}{lrcl} x^2: & 0 & = & A + C \\ x: & 0 & = & B + 2C \\ \text{konst}: & 1 & = & -A - B + C \end{array}\right\}
\quad
\begin{array}{ccc|c} A & B & C & \\ \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ -1 & -1 & 1 & 1 \end{array}\right)$$

test av felaktig ansats vid "dubbelrot"

$$\frac{1}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)} = \frac{(A+B)}{x+1}$$

$$= \frac{A(x+1) + B(x+1)}{(x+1)^2}$$

$$1 = Ax + A + Bx + B$$

$$\left.\begin{array}{ll} x: & 0 = A + B \\ \text{konst}: & 1 = A + B \end{array}\right\}$$

går ej!

Om ekv. syst. inte har entydig lösning är det fel i ansatsen.

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korrekt ansats vid "dubbelrot", (faktor med multiplicitet 2)

$$\frac{1}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{A(x+1) + B}{(x+1)^2}$$

1:a grads uttryck

2:a grads

— — — — —

Ansats vid "trippelrot"

$$\frac{1}{(x-a)^3} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3}$$