$\boxed{F16}$ Medelvärdessatsen (till integralkalkylen)
Analysens huvudsats (kap 6)

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ex) $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx$ $\quad \left\{ \begin{array}{l} \boxed{x = \tan(t)} \\ \frac{dx}{dt} = 1+\tan^2(t) = \frac{1}{\cos^2(t)} \\ dx = \frac{1}{\cos^2(t)}\,dt \end{array} \right.$

Test av
$\cancel{t = \sqrt{1+x^2}}$
$\cancel{t^2 - 1 = x^2}$
$\cancel{x = \sqrt{t^2-1}}$
$\cancel{dx = \frac{1}{2\sqrt{t^2-1}}\,2t}$
$\cancel{\int \frac{1}{t} \cdot \frac{t}{\sqrt{t^2-1}}\,dt}$ inte bättre

$$= \int \frac{1}{\sqrt{1+\tan^2 t}} \cdot \frac{1}{\cos^2(t)}\,dt$$

$$= \int \frac{1}{\sqrt{\frac{1}{\cos^2 t}}}\,\frac{1}{\cos^2(t)}\,dt = \int |\cos t| \cdot \frac{1}{\cos^2(t)}\,dt = \left\{ \begin{array}{l} \text{Antag} \\ \cos t > 0 \end{array} \right\}$$

$$= \int \frac{\overbrace{\cos(t)\,dt}^{du}}{\underbrace{1-\sin^2(t)}_{1-u^2}} = \left\{ \begin{array}{l} \boxed{u = \sin t} \\ \frac{du}{dt} = \cos t \\ du = \cos(t)\,dt \end{array} \right\}$$

$$= \int \frac{du}{1-u^2} = \int \frac{du}{(1-u)(1+u)}$$

Partialbråk:

$$\text{VL} \quad \frac{1}{(1-u)(1+u)} = \frac{A}{1-u} + \frac{B}{1+u} = \frac{A(1+u)+B(1-u)}{(1-u)(1+u)} \quad \text{HL}$$

Identifiera tälj.

$$\left.\begin{array}{lcl} u: & 0 = & A - B \\ u^0: & 1 = & A + B \end{array}\right\} \quad \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 1 \end{array}\right) \sim \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 1 \end{array}\right) \end{array}$$

$B = 1/2 = A$

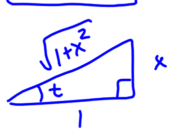
$$I = \text{forts} = \int \frac{1}{(1-u)(1+u)}\,du = \int \frac{1/2}{1-u} + \frac{1/2}{1+u}\,du$$

$$= \frac{1}{2}\underbrace{\ln|1-u|}_{-1} + \frac{1}{2}\ln|1+u| + C = \frac{1}{2}\ln\left|\frac{1+\sin t}{1-\sin t}\right| + C$$

inre derivata till

byt tillbaka $u = \sin t$

där $\boxed{x = \tan t}$



$\frac{x}{1} = \tan(t)$

ur fig. $\sin(t) = \frac{x}{\sqrt{1+x^2}}$

$$= \frac{1}{2}\ln\left|\frac{1+\frac{x}{\sqrt{1+x^2}}}{1-\frac{x}{\sqrt{1+x^2}}}\right| + C =$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right| + C =$$

$$= \left\{ \begin{array}{l} \text{förläng} \\ \text{med} \\ \text{nämnarens} \\ \text{konjugat} \end{array} \right\} = \frac{1}{2}\ln\left|\frac{(\sqrt{1+x^2}+x)^2}{1+x^2-x^2}\right| + C = \frac{1}{\cancel{2}} \cdot \cancel{2}\ln\left|\sqrt{1+x^2}+x\right| + C$$

$$\int \frac{x+2}{x^2+2x+2}\,dx$$

a) faktorisera N

$x^2+2x+2=0$

$x=-1\pm\underbrace{\sqrt{1-2}}_{<0}$

Nej

$$\bullet \int \frac{u'(x)}{u(x)}\,dx = \ln|u(x)| + C$$

b) kvadratkomp N

$(x+1)^2 \cancel{+}\ 1$

Ja!

$$\bullet \int \frac{1}{x^2+1}\,dx = \arctan(x) + C$$

(nämnaren)

Sätt $t = x^2+2x+2$

$\frac{dt}{dx} = 2x+2$

$\frac{dt}{dx} = 2(x+1)$

$\frac{dt}{2} = (x+1)\,dx$

$$\int \frac{x+1+1}{x^2+2x+2}\,dx = \int\left(\frac{x+1}{x^2+2x+2} + \frac{1}{x^2+2x+2}\right)dx$$

$$\frac{1}{2}\int \frac{dt}{t} + \int \frac{1}{(x+1)^2+1}\,dx$$

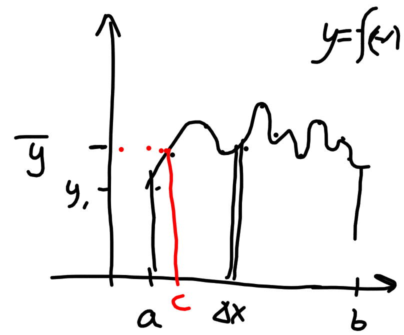
$$= \frac{1}{2}\ln|t| + \arctan(x+1) + C$$

$$= \frac{1}{2}\ln|x^2+2x+2) + \arctan(x+1) + C$$

Kap 6 Bestämda integraler

<u>Integralkalkylens medelvärdessats.</u>

$$\bar{y} = \frac{y_1 + y_2 + \ldots + y_n}{n} = \frac{\sum_{i=1}^{n} y_i}{n}$$



Dela in x-intervallet $[a,b]$ i n st delar med bredden Δx

$$\frac{b-a}{n} = \Delta x \iff n = \frac{b-a}{\Delta x}$$

$$\bar{y} = \frac{\sum y_i}{n} = \longrightarrow \frac{\int f(x)\,dx}{b-a}$$

då $n \to \infty$ och $\Delta x \to 0$

$$\boxed{\bar{f}(c) = \frac{1}{b-a} \cdot \int_a^b f(x)\,dx \quad \text{för något } c, \quad a \le c \le b}$$

$\bar{f}$ är medelvärdet av funktionen $f(x)$ i $a \le x \le b$

ex) $y = f(x) = 12 - 3x \quad ; \quad -1 \le x \le 4$

Sök medelvärdet av y i intervallet.

$$\bar{y} = \frac{1}{b-a} \cdot \int_a^b f(x)\,dx$$

$$= \frac{1}{4-(-1)} \cdot \int_{-1}^{4} (12-3x)\,dx = \frac{1}{5} \cdot \left[12x - \frac{3x^2}{2}\right]_{-1}^{4} =$$

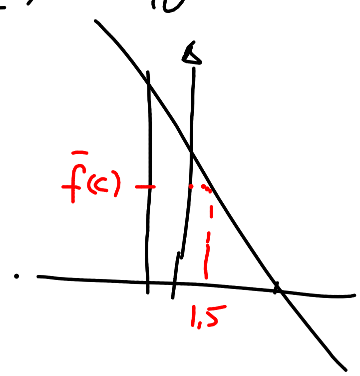
$$= \frac{1}{5}\left[12 \cdot 4 - \frac{3 \cdot 4^2}{2} - \left(12(-1) - \frac{3 \cdot (-1)^2}{2}\right)\right] =$$

$$= \frac{1}{5}\left(48 - 24 + 12 + \frac{3}{2}\right) = \frac{75}{10} = 7{,}5$$

$\bar{f}(c) = 7{,}5 \qquad c = ?$

$12 - 3x = 7{,}5$

$x = 1{,}5$



Integralkalkylens huvudsats

Om $A(x) = \int_a^x f(t)\,dt$ gäller $A'(x) = f(x)$

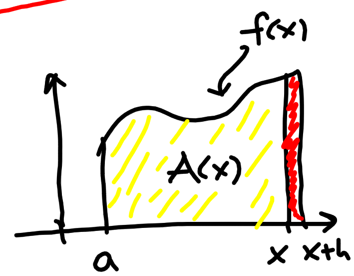
och därefter:

$$\boxed{\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)}$$

- - - -

Låt $A(x) = \int_a^x f(t)\,dt$

(= Arean under kurvan om $f(x) \geq 0$)



$$A'(x) = \left\{ \begin{matrix} \text{derivatans} \\ \text{definition} \end{matrix} \right\} = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

{ $A(x+h) - A(x)$ är röda areaskivan med bredd h markerad i figuren: }

Medelv.satsen för integraler.

$$A(x+h) - A(x) = \int_x^{x+h} f(t)\,dt = f(c) \cdot \underbrace{(x+h-x)}_{=h}$$

för något c där $c \in [x, x+h]$.

Då $h \to 0$ gäller att $c \to x$.

$$\because \; A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} =$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot f(c) \cdot h = \lim_{\substack{h \to 0 \\ c \to x}} f(c) = f(x).$$

(1) $A(x) = \int_a^x f(t)\,dt \Rightarrow A'(x) = f(x)$

även : $F'(x) = f(x)$ där $F(x)$ primitiv fkn. till $f(x)$.

Då $A(x)$ och $F(x)$ har samma derivata gäller:

(2) $A(x) = F(x) + C$

Det kan bara skilja en konstant mellan funktionerna som har samma derivata.

Ins. $\underline{x=a}$ i (1) ger

$$A(a) = \int_a^a f(t)\,dt = 0$$

ins i (2) ger $0 = F(a) + C \Leftrightarrow$

$C = -F(a)$

$$\therefore A(x) = F(x) - F(a)$$

$$\therefore A(x) = \int_a^x f(t)\,dt = F(x) - F(a)$$

Ins. $\underline{x=b}$ ger slutligen :

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

$$= \Big[F(x) + C\Big]_a^b = \Big(F(b) + \cancel{C}\Big) - \Big(F(a) + \cancel{C}\Big)$$

$\therefore$ inget C behövs vid. bestämda integraler.

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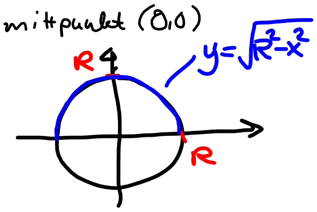
ex) Area av $\frac{1}{4}$ cirkel.

$x^2 + y^2 = R^2$ cirkelns ekv. med radie R mittpunkt $(0,0)$

$y = \pm\sqrt{R^2 - x^2}$



$$A = \int_0^R f(x)\,dx$$

$$= \int_0^R \sqrt{R^2 - x^2}\,dx = \left\{ \begin{array}{l} x = R\cdot\sin t \\ x^2 = R^2\cdot\sin^2 t \\ dx = R\cdot\cos t\,dt \end{array} \right.$$

gränser: $x = 0 \Rightarrow t = 0$

$x = R \Rightarrow R = R\cdot\sin t$, $1 = \sin t$, $t = \frac{\pi}{2}$

$x: [0, R] \to t\,[0, \frac{\pi}{2}]$

$$= \int_0^{\pi/2} \sqrt{R^2 - R^2\cdot\sin^2 t}\cdot R\cdot\cos t\,dt$$

$$= \int_0^{\pi/2} R\underbrace{\sqrt{1 - \sin^2 t}}_{|\cos t| \geq 0 \text{ för } 0 \leq t \leq \frac{\pi}{2}}\cdot R\cdot\cos t\,dt$$

jämn exponent $\longrightarrow$ "cos dubbelvinkeln"

$$= \int_0^{\pi/2} R^2\cos^2 t\,dt = R^2\cdot\int_0^{\pi/2} \frac{1 + \cos 2t}{2}\,dt$$

$$\left\{ \begin{array}{l} \cos^2 t + \sin^2 t = 1 \\ \cos^2 t - \sin^2 t = \cos 2t \\ \hline 2\cos^2 t = 1 + \cos 2t \end{array} \right.$$

$$= \frac{R^2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{\pi/2}$$

$$= \frac{R^2}{2}\left[\frac{\pi}{2} + \underbrace{\frac{\sin 2\frac{\pi}{2}}{2}}_{=0} - \big(\underbrace{0 + \sin 0}_{=0}\big)\right]$$

$$= \frac{\pi}{4}R^2$$

Känt:

Area av cirkel $= \pi R^2$

$\therefore$ –"– $\frac{1}{4}$ –"– $= \frac{1}{4}\pi R^2$. Ja!