$\boxed{F17}$ Area, Båglängd

FN.
6.11 b)

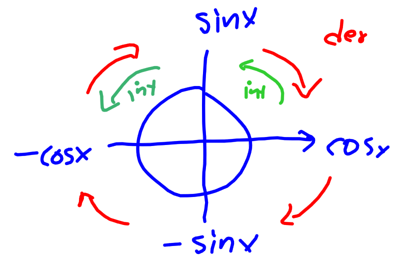
$$\int_0^{\pi/4} \frac{\sin^5 x}{\cos^7 x}\,dx = \int_0^{\pi/4} \frac{1}{\cos^2 x}\cdot\frac{\sin^5 x}{\cos^5 x}\,dx$$

$$= \int_0^{\pi/4} \underbrace{\frac{1}{\cos^2 x}\cdot(\tan x)^5\,dx}_{dt} = \left\{\begin{array}{l} \boxed{t = \tan x} \\ dt = \dfrac{1}{\cos^2 x}\,dx \\ x = 0 \Rightarrow t = \tan 0 = 0 \\ x = \frac{\pi}{4} \Rightarrow t = \tan\frac{\pi}{4} = 1 \end{array}\right.$$

$$= \int_0^1 t^5\,dt = \left[\frac{t^6}{6}\right]_0^1 = \frac{1^6}{6} - \frac{0^6}{6} = \frac{1}{6}$$

Ex) Bestäm medelvärdet av $f(x) = e^{-x} + \cos x$ på intervallet $[-\frac{\pi}{2}, 0]$.

$$\bar{f} = \frac{1}{b-a}\cdot\int_a^b f(x)\,dx$$



$$\bar{f} = \frac{1}{0-(-\frac{\pi}{2})}\int_{-\pi/2}^{0} (e^{-x} + \cos x)\,dx$$

$$= \frac{2}{\pi}\cdot\left[-e^{-x} + \sin x\right]_{-\frac{\pi}{2}}^{0} =$$

$$= \frac{2}{\pi}\Big[\underbrace{-e^{-0}}_{-1} + \underbrace{\sin 0}_{=0} - \big(\underbrace{-e^{-(-\frac{\pi}{2})}}_{+e^{\frac{\pi}{2}}} + \underbrace{\sin(-\tfrac{\pi}{2})}_{-1}\big)\Big] =$$

$$= \frac{2}{\pi}\left[-1 + e^{\pi/2} + 1\right] = \frac{2}{\pi}\cdot e^{\pi/2}$$

6.11(b) Alt.

$$\int_0^{\pi/4} \frac{\sin^5 x}{\cos^7 x}\,dx = \int_0^{\pi/4} \frac{\sin x \cdot (1-\cos^2 x)^2}{\cos^7 x}\,dx =$$

i.d.

$$= \int_0^{\pi/4} \frac{\sin x\,(1 - 2\cdot\cos^2 x + \cos^4 x)}{\cos^7 x}\,dx$$

$$= \int_0^{\pi/4} \left(\frac{\sin x}{\cos^7 x} - \frac{2\cdot \sin x}{\cos^5 x} + \frac{\sin x}{\cos^3 x} \right) dx$$

$$\left\{ \begin{array}{l} t = \cos x \\ \frac{dt}{dx} = -\sin x \\ x = 0 \Rightarrow t = \cos 0 = 1 \\ x = \frac{\pi}{4} \Rightarrow t = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \end{array} \right\}$$

$$= \int_1^{1/\sqrt{2}} \left(\frac{-1}{t^7} + \frac{2}{t^5} - \frac{1}{t^3} \right) dt$$

$$= \int_1^{1/\sqrt{2}} -t^{-7} + 2t^{-5} - t^{-3}\,dt$$

$$= \left[\frac{-t^{-6}}{-6} + 2\frac{t^{-4}}{-4} - \frac{t^{-2}}{-2} \right]_1^{1/\sqrt{2}}$$

$$= \frac{1}{6}\left(\frac{1}{\sqrt{2}}\right)^{-6} - \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^{-4} + \left(\frac{1}{\sqrt{2}}\right)^{-2} - \left(\frac{1}{6} - \frac{1}{2} + \frac{1}{2}\right) =$$

$$= \frac{1}{6}(\sqrt{2})^6 - \frac{1}{2}(\sqrt{2})^4 + (\sqrt{2})^2 - \frac{1}{6} =$$

$$= \frac{1}{6}\,2^3 - \frac{1}{2}\,2^2 + 2 - \frac{1}{6} = \underline{\underline{\frac{7}{6}}}$$

Obestämda integraler = Primitiva funktioner

$$\int f(x)\,dx = F(x) + C$$

$$\int f(t)\,dt = F(t) + C$$

olika funktioner

Bestämda integraler ger <u>tal</u>.

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

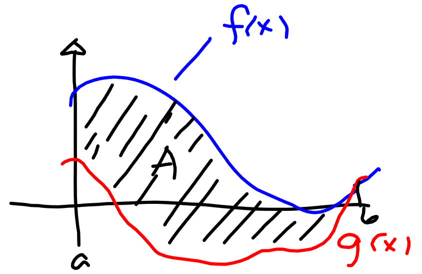
$$\int_a^b f(t)\,dt = \Big[F(t)\Big]_a^b = F(b) - F(a)$$

Area

Arean mellan två kontinuerliga funktioner mellan $a \leq x \leq b$.

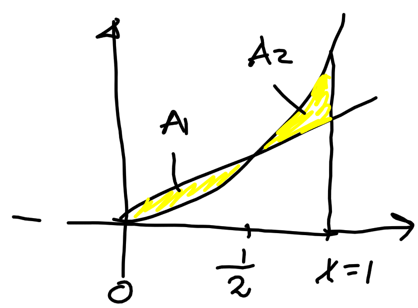
$$A = \int_a^b (f(x) - g(x))\,dx$$

där $f(x) \geq g(x)$ i hela intervallet.



ex) tenta mars-13.

Area mellan $y = x^3$ och linjerna $y = x/4$ och $x = 1$



- skärningspunkter: $y = y$

$$x^3 = \frac{x}{4}$$

$$x^3 - \frac{x}{4} = 0$$

$$x\left(x^2 - \frac{1}{4}\right) = 0$$

$x = 0$ eller $x^2 = \frac{1}{4}$

$$x = (\pm)\frac{1}{2}$$

$$A_1 = \int_0^{1/2} \left(\frac{x}{4} - x^3\right) dx = \left[\frac{1}{4} \cdot \frac{x^2}{2} - \frac{x^4}{4}\right]_0^{1/2} =$$

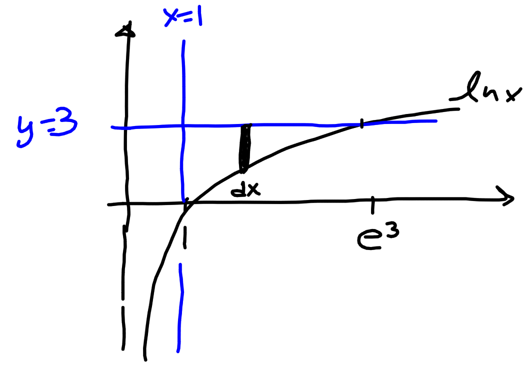
$$= \frac{1}{8}\left(\frac{1}{2}\right)^2 - \frac{1}{4} \cdot \left(\frac{1}{2}\right)^4 - \left(\frac{1}{8} \cdot 0 - 0\right) = \frac{1}{8} \cdot \frac{1}{4} - \frac{1}{4} \cdot \frac{1}{16} = \underline{\underline{\frac{1}{64}}}$$

$$A_2 = \int_{1/2}^{1} \left(x^3 - \frac{x}{4}\right) dx = \ldots = \frac{9}{64}$$

Svar Area $= A_1 + A_2 = \frac{10}{64} = \underline{\underline{\frac{5}{32}}}$.

ex) Arean mellan:

$$\begin{cases} y = \ln x \\ x = 1 \\ y = 3 \end{cases}$$



• skärningspunkter:

$y = y$

$\ln x = 3$

$x = e^3$ då $y = 3$

$dA =$ höjd: $3 - \ln x$, dx

dx-integral:

$$A = \int_1^{e^3} \underbrace{(3 - \ln x)}_{\text{övre} - \text{undre}} \, dx = \underbrace{\left[3x\right]_1^{e^3}}_{I_1} - \underbrace{\int_1^{e^3} 1 \cdot \ln x \, dx}_{I_2}$$

$$I_2 = \int_1^{e^3} 1 \cdot \ln x \, dx \quad \text{int} \quad \text{der}$$

$$= \left[x \cdot \ln x\right]_1^{e^3} - \int_1^{e^3} x \cdot \frac{1}{x} \, dx$$

$$= \left[x \cdot \ln x\right]_1^{e^3} - \int_1^{e^3} 1 \, dx$$

$$= \left[x \cdot \ln x - x\right]_1^{e^3} =$$

$$= e^3 \cdot \underbrace{\ln e^3} - e^3 - (1 \cdot \underbrace{\ln 1}_{=0} - 1)$$

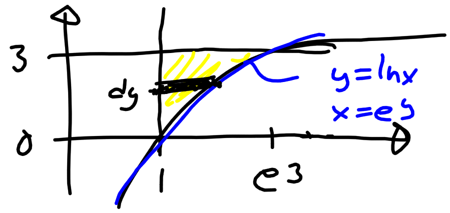
$$= e^3 \cdot 3 - e^3 + 1$$

$$= \underline{2e^3 + 1}$$

$$I_1 = \left[3x\right]_1^{e^3} = 3e^3 - (3 \cdot 1) = 3e^3 - 3$$

$$A = I_1 - I_2 = 3e^3 - 3 - (2e^3 + 1) = \underline{\underline{e^3 - 4}}$$

Area mellan $\begin{cases} y = \ln x \iff x = e^y \\ y = 3 \\ x = 1 \end{cases}$



dy-integral:

skärningspunkt:

$x=1$ ger $y = \ln 1 = 0$

$y = 3$

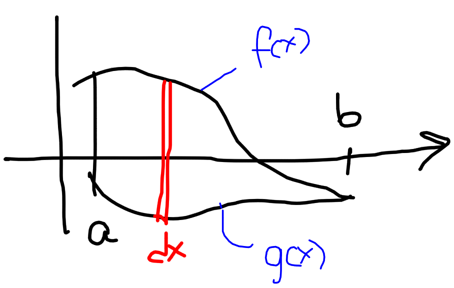
$$A = \int dA = \int_0^3 (e^y - 1)\,dy = \left[e^y - y\right]_0^3 = e^3 - 3 - (e^0 - 0$$

$$= \underline{\underline{e^3 - 4}}$$

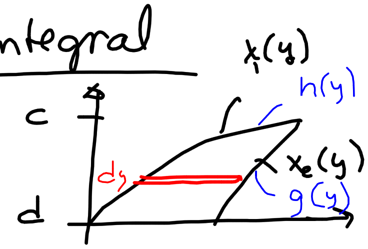
dx-integral:

$$A = \int_{x=a}^{x=b} \left(f(x) - g(x)\right) dx$$

där $f(x) \geq g(x)$ för $x \in [a,b]$



dy-integral



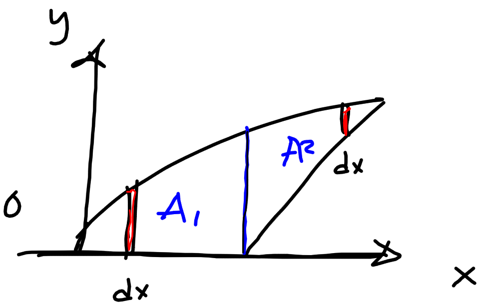
$$A = \int_d^c \left(x_2(y) - x_1(y)\right) dy$$

$$A = \int_{y=d}^{y=c} \left(g(y) - h(y)\right) dy$$

där $g(y) \geq h(y)$

för $y \in [d,c]$

Area mellan $\begin{cases} x = y^{3/2} \\ x = y+4 \\ x\text{-axeln} \Leftrightarrow y=0 \end{cases}$



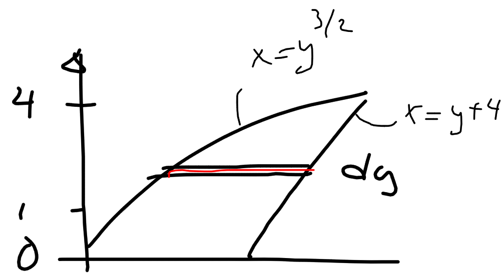
• skärningspunkter:

$$y^{3/2} = y+4$$

$$(y^{3/2})^2 = (y+4)^2$$

$$y^3 = y^2 + 8y + 16$$

$$y^3 - y^2 - 8y - 16 = 0$$



Gissa rot: ± 1 ± 16 ± 2 ± 8 ± 4

$y=4$ ger $4^3 - 4^2 - 8.4 - 16 = 0$

polynomdiv: ger $(y-4)\underbrace{(y^2 + 3y + 4)}_{\text{saknar reella nollställen}} = 0$

dy-integral:

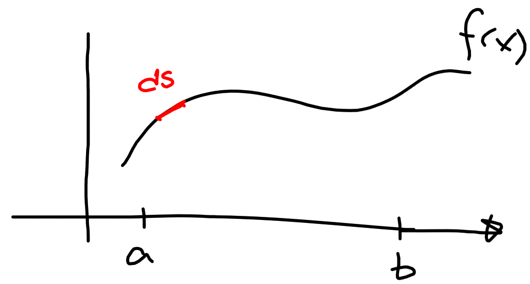
$$A = \int_0^4 (y + 4 - y^{3/2})\, dy = \left[\frac{y^2}{2} + 4y - \frac{y^{5/2}}{5/2} \right]_0^4$$

$$= \frac{4^2}{2} + 4.4 - 4^{5/2} \cdot \frac{2}{5} - (0 + 0 - 0) =$$

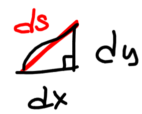
$$= 8 + 16 - (\underbrace{4^{1/2}}_{2})^5 \cdot \frac{2}{5} = 8 + 16 - 32 \cdot \frac{2}{5} = \ldots = \underline{\underline{\frac{56}{5}}}$$

Båglängd:

Sträckan $s = \int ds$



$$ds = \sqrt{dx^2 + dy^2} \quad \text{pyth.}$$



$$= \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right) \cdot dx^2}$$

$$= \sqrt{1 + (y'(x))^2}\, dx$$

$$\boxed{\text{Båglängd:} \quad s = \int_a^b \sqrt{1 + (y'(x))^2}\, dx}$$

ex) $y = \frac{x^2}{4} - \frac{\ln x}{2} \qquad 1 \leq x \leq 4$

Bestäm båglängden.

$$y'(x) = \frac{2 \cdot x}{4} - \frac{1}{2} \cdot \frac{1}{x} = \frac{1}{2}\left(x - \frac{1}{x}\right)$$

$$(y')^2 = \frac{1}{4}\left(x - \frac{1}{x}\right)^2 = \frac{1}{4}\left(x^2 - 2 + \frac{1}{x^2}\right) \quad (*)$$

$$s = \int_1^4 \sqrt{1 + (y')^2}\, dx = \int_1^4 \sqrt{1 + \frac{1}{4}\left(x^2 - 2 + \frac{1}{x^2}\right)}\, dx = \ldots$$

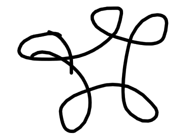
$$= \int_1^4 \sqrt{\frac{1}{4}\left(x^2 + 2 + \frac{1}{x^2}\right)}\, dx = \int_1^4 \sqrt{\frac{1}{4}\left(x + \frac{1}{x}\right)^2}\, dx$$

jfr m. (*)

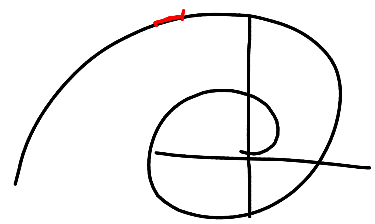
$$= \int_1^4 \frac{1}{2}\left(x + \frac{1}{x}\right) dx = \frac{1}{2}\left[\frac{x^2}{2} + \ln x\right]_1^4 =$$

$$= \frac{1}{2}\left[8 + \ln 4 - \left(\frac{1}{2} + \ln 1\right)\right] = \ldots$$

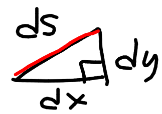
ex) Båglängd på parameterform



$$\begin{cases} x = 3t^2 \\ y = 2t^3 \end{cases} \qquad 0 \le t \le 2$$



$S = \int ds$



$\boxed{ds} = \sqrt{dx^2 + dy^2}$ pyth.

$$= \sqrt{(dx^2 + dy^2)\frac{dt^2}{dt^2}} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$\boxed{= \sqrt{x'(t)^2 + y'(t)^2}\,dt}$$

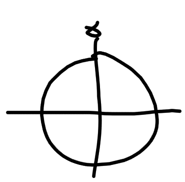
$x'(t) = 6t$
$y'(t) = 6t^2$

$$S = \int_0^2 \sqrt{(6t)^2 + (6t^2)^2}\,dt = \int_0^2 \sqrt{36t^2(1+t^2)}\,dt$$

$$= \int_0^2 6 \cdot t \cdot \sqrt{1+t^2}\,dt = \begin{cases} u = 1+t^2 \\ du = 2t\,dt \\ t = 0 \Rightarrow u = 1+0 = 1 \\ t = 2 \Rightarrow u = 1+2^2 = 5 \end{cases}$$

$$= \int_1^5 3 \cdot \sqrt{u}\,du = \left[3 \cdot \frac{u^{3/2}}{3/2}\right]_1^5 = \underline{\underline{2 \cdot (5^{3/2} - 1)}}$$

ex)



$x^2 + y^2 = 1$ Båglängd ?