$\boxed{F18}$ Rotationsvolymer

Rep.) Area mellan $y=1$ och $y=\cos^2 x$

- skärningspunkter:

$$y = y$$
$$1 = \cos^2 x$$
$$\cos x = \pm 1$$

$\cos x = 1$, $\underline{x = 0}$ $(+2\pi n)$ eller $\cos x = -1$, $\underline{x = \pi}$ $(+2\pi n)$

- övre funktion? $y\left(\frac{\pi}{2}\right) = 1$, $y\left(\frac{\pi}{2}\right) = \left(\cos\left(\frac{\pi}{2}\right)\right)^2 = 0$

sätt in ett x-värde mellan gränserna och jämför.

$\{y=1\} \geq \{y=\cos^2 x\}$ i intervallet.

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$$A = \int_0^{\pi} (1-\cos^2 x)\,dx = \int_0^{\pi} \sin^2 x\,dx$$

$$= \int_0^{\pi} \left(\frac{1-\cos(2x)}{2}\right) dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}\left[\pi - 0 - (0-0)\right] = \underline{\underline{\frac{\pi}{2}}}$$

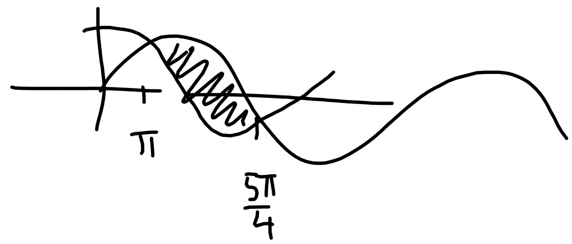
alt. $\left\{ \begin{array}{l} \boxed{t = 2x} \\ dt = 2\,dx \iff \frac{dt}{2} = dx \\ x = [0,\pi] \to t = [0, 2\pi] \end{array} \right\}$

$$= \int_0^{2\pi} \frac{1-\cos t}{2} \cdot \frac{dt}{2} = \frac{1}{4}\left[t - \sin t\right]_0^{2\pi} =$$

$$= \frac{1}{4}\left[2\pi - \underbrace{\sin 2\pi}_{0} - (0 - \sin 0)\right] = \underline{\underline{\frac{\pi}{2}}}$$

Area mellan:

$$\begin{cases} y = \sin x \\ y = \cos x \end{cases}$$



• skärn.pkt: $\sin x = \cos x$

$$\frac{\sin x}{\cos x} = 1$$

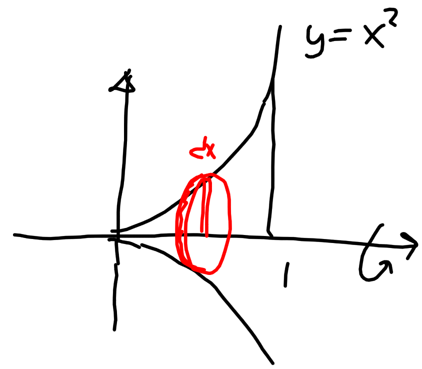
$$\tan x = 1$$

$$x = \frac{\pi}{4} + \pi \cdot n$$

$$x = \frac{\pi}{4} \qquad x = \frac{5\pi}{4}$$

$$A = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx = \ldots$$

Rotationsvolymer



Skivmetoden

ex) $y = x^2$ $\quad 0 \le x \le 1$

roter**a** kring x-axeln.

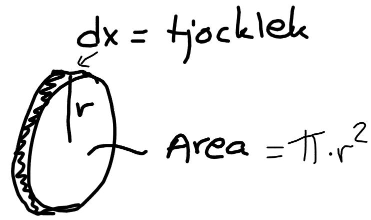
Bestäm rotationsvolymen.

$$\boxed{dV} = \text{volym av tunn cirkelskiva}$$

$$= \text{Area} \cdot \text{tjocklek}$$

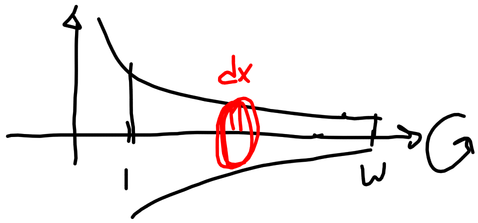
$$= \pi r^2 \cdot dx$$

$$= \pi y^2 \cdot dx \quad = \pi [f(x)]^2 dx$$



$$V = \int_0^1 dV = \int_0^1 \pi f(x)^2 dx = \int_0^1 \pi (x^2)^2 dx = \left[\pi \frac{x^5}{5}\right]_0^1 = \frac{\pi}{5}$$

Ex) Volymen som genereras då $f(x) = \frac{1}{x}$ roterar kring <u>x-axeln</u> för $1 \leq x \leq \omega$.



$$dV = A \cdot dx$$
$$= \pi r^2 \cdot dx$$
$$= \pi \left(\frac{1}{x}\right)^2 dx$$

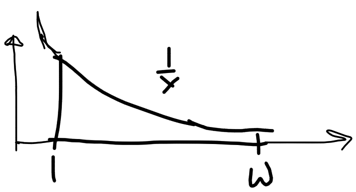
$$V = \int_1^\omega \pi \cdot \frac{1}{x^2}\, dx = \int_1^\omega \pi \cdot x^{-2}\, dx = \left[\pi \frac{x^{-1}}{-1}\right]_1^\omega$$

$$= \pi \cdot \left(\frac{-1}{\omega}\right) - \pi \cdot (-1) = \underline{\underline{\pi\left(1 - \frac{1}{\omega}\right)}}$$

• $\lim_{\omega \to \infty} V = \lim_{\omega \to \infty} \pi\left(1 - \frac{1}{\omega}\right) = \underline{\underline{\pi}}$ Ändlig volym.

— — —

Area :

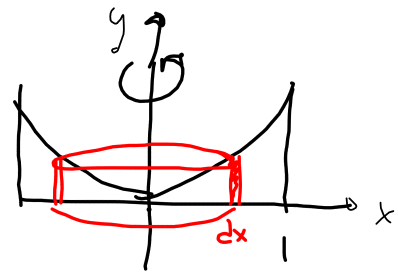


$$A = \int_1^\omega \left(\frac{1}{x} - 0\right) dx = \Big[\ln|x|\Big]_1^\omega = \ln|\omega| - \underbrace{\ln 1}_{=0} = \ln|\omega|$$

$$\lim_{\omega \to \infty} A = \infty .$$

Oändlig area men ändlig volym.

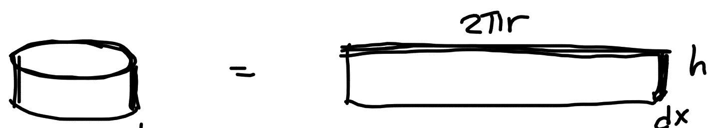
Rörmetoden (cylinderskalmetoden)



ex) $y = x^2$ och x-axeln $0 \le x \le 1$

Rotera kring <u>y-axeln.</u>

$\boxed{dV}$ = volym av tunnt rör



= omkrets · höjd · tjocklek

$= \boxed{2\pi r \cdot h \cdot dx}$ =

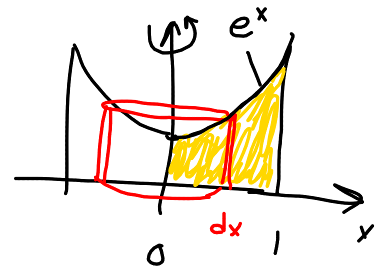
$= 2\pi x \cdot y \cdot dx = 2\pi x \cdot f(x)\, dx.$

$$V = \int dV = \int 2\pi x \cdot f(x)\; dx$$

Rörmetoden: Vid rot. kring <u>y-axeln.</u>

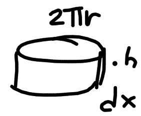
$$V = \int_0^1 2\pi x \cdot x^2\, dx = 2\pi \int_0^1 x^3 dx = 2\pi \left[\frac{x^4}{4}\right]_0^1 = 2\pi \cdot \frac{1}{4} = \underline{\frac{\pi}{2}}$$

ex) Området $0 \le y \le e^x$, $0 \le x \le 1$ roterar kring <u>y -axeln</u>.



<u>Rörmetoden</u> :

dV = tunnt rör



$$= 2\pi r \cdot h \cdot dx$$

$$= 2\pi x \cdot y \cdot dx \quad = \quad 2\pi x \cdot e^x \, dx$$

$$V = \int_0^1 2\pi x \cdot e^x \, dx \quad = \quad \text{P.I}$$

= Partiell integration

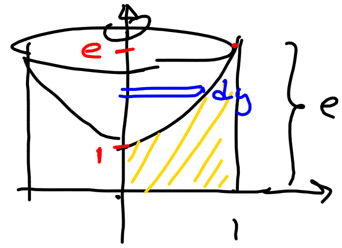
$$= 2\pi \left[x \cdot e^x \right]_0^1 \; - \; 2\pi \int_0^1 1 \cdot e^x \, dx$$

$$= 2\pi \left[1 \cdot e^1 - 0 \right] \; - \; 2\pi \cdot \left[e^x \right]_0^1$$

$$= 2\pi e \; - \; 2\pi \left(e - \underbrace{e^0}_{=1} \right) \quad = \quad \cancel{2\pi e} - \cancel{2\pi e} + \underline{\underline{2\pi}}$$

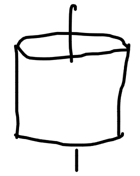
Ex) $0 \le y \le e^x \qquad 0 \le x \le 1$

rotera kring y-axeln



Volym = hel (kompakt) cylinder − volym av skål

$V_1 \qquad - \qquad V_2$



- $V_1 = B \cdot h = \pi \cdot \text{basradie}^2 \cdot \text{ytterhöjd} = \pi \cdot 1^2 \cdot e = \underline{\underline{\pi e}}$

- V_2: skål (volym av urgröpning)

dV_2 = tunn cirkelskiva

$$= A \cdot dy = \pi r^2 \cdot dy$$
$$= \pi \cdot x^2\, dy = \pi (\ln y)^2\, dy$$
$$= \pi \ln^2 y\, dy$$

$y = e^x \iff x = \ln(y)$

$$V_2 = \int_1^e \pi (\ln y)^2\, dy \overset{\text{P.I}}{=} \pi \left[y \cdot (\ln y)^2 \right]_1^e - \pi \cdot \int_1^e \cancel{y} \cdot 2 \cdot \ln y \cdot \frac{1}{\cancel{y}}\, dy$$

$$= \pi \left[e \underbrace{(\ln e)^2}_{=1} - 1 \cdot \underbrace{(\ln 1)^2}_{0} \right] - \pi 2 \cdot \int_1^e 1 \cdot \ln y\, dy$$

$$= \pi \cdot e - 2\pi \left(\left[y \cdot \ln y \right]_1^e - \int_1^e y \cdot \frac{1}{y}\, dy \right)$$

$$= \pi \cdot e - 2\pi \left[y \cdot \ln y - y \right]_1^e =$$

$$= \pi \cdot e - 2\pi \left[e \cdot \underbrace{\ln e}_{1} - e - (1 \cdot \underbrace{\ln 1}_{=0} - 1) \right]$$

$$= \underline{\pi \cdot e - 2\pi}$$

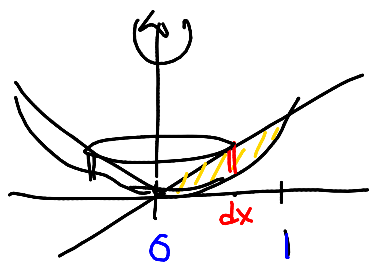
$\therefore \; V = V_1 - V_2 = \pi e - (\pi e - 2\pi) = \underline{\underline{2\pi}}$

ex) $\left.\begin{array}{l} y = x \\ y = x^2 \end{array}\right\}$ området roterar kring $\underline{y\text{-axeln}}$

, skärningspunkter : $x = x^2$

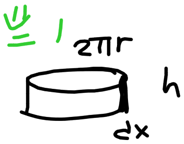
$x - x^2 = 0$

$x(1-x) = 0$



$x = 0$ eller $x = 1$

$y = 0$ $y = 1$



$\boxed{\text{Rörmetoden}}$:

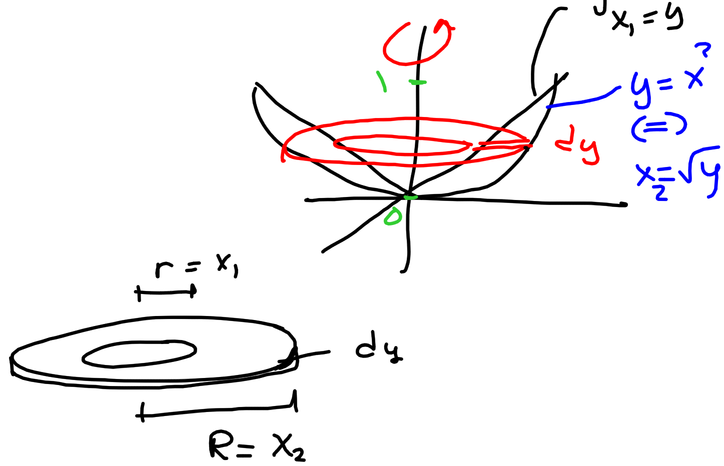
$dV = 2\pi r \cdot h \cdot dx$

$= 2\pi \cdot x \cdot (x - x^2)\, dx$

$$V = \int_0^1 2\pi (x^2 - x^3)\, dx = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = 2\pi \left[\frac{1}{3} - \frac{1}{4} - (0-0) \right]$$

$$= 2\pi \left(\frac{4-3}{12} \right) = \underline{\underline{\frac{\pi}{6}}}$$

Alt. $\boxed{\text{Skivmetoden}}$



$dV =$ ihålig cirkelskiva

$= (\pi R^2 - \pi r^2) \cdot dy$

$= (\pi (x_2)^2 - \pi (x_1)^2)\, dy$

$= (\pi (\sqrt{y})^2 - \pi y^2)\, dy$

$$V = \pi \cdot \int_0^1 (y - y^2)\, dy = \pi \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^1 = \pi \left[\frac{1}{2} - \frac{1}{3} - (0-0) \right]$$

$$= \pi \left[\frac{3-2}{6} \right] = \underline{\underline{\frac{\pi}{6}}}$$

Ex)

$$\left.\begin{array}{l} y=\sqrt{x} \\ y=1 \\ x=4 \end{array}\right\} \text{ rotera kring } \underline{\underline{y=1}}$$

Bestäm rotationsvolymen.

Svar: $\frac{7\pi}{6}$