F8 Gausselimination, Determinanter

Rep.
Gauss-elimination

$$\begin{cases} x - 2y + z = 8 \\ 2x - 3y + 4z = 20 \\ 3x - 8y + z = 22 \end{cases}$$

I matrisform:

$$\underbrace{\begin{pmatrix} 1 & -2 & 1 \\ 2 & -3 & 4 \\ 3 & -8 & 1 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\bar{x}} = \underbrace{\begin{pmatrix} 8 \\ 20 \\ 22 \end{pmatrix}}_{B}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 2 & -3 & 4 & 20 \\ 3 & -8 & 1 & 22 \end{array}\right) \cdot(-2)\ (-3) \sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 0 & 1 & 2 & 4 \\ 0 & -2 & -2 & -2 \end{array}\right) \cdot(2) \sim$$

$$\sim \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 2 & 6 \end{array}\right) \end{array}$$

Nu trappstegsform (triangulär form)
- Bakåtsubstitution ger sedan lösningen...

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Fortsätt "göra" nollor ovanför pivotelementen (som ska vara 1), Börja nerifrån.

"Gauss-Jordan" =
= reducerad trappstegsform
= reduced row echelon form (rref)

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right) (-2)(-1) \sim \left(\begin{array}{ccc|c} 1 & -2 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right) (2) \sim$$

$$\sim \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right) \end{array} \qquad \underbrace{}_{I} \ \underbrace{}_{\bar{x}}$$

Svar: $\bar{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

$(x, y, z) = (1, -2, 3)$

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$$A\bar{x} = B$$

total-matris: $(A \mid B) \sim \ldots \sim (I \mid \bar{x})$ Gauss-Jordan

ex)
$$\begin{array}{c} \begin{array}{ccccc} x & y & z & u & v \end{array} \\ \left(\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right) \end{array}$$

y, u, v fria variabler.

Sätt $y = t$, $u = s$, $v = r$ parametrar.

$$\sim \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right)$$

rad2: $z = 1 - v = 1 - r$

rad1: $x + 2t + s - 3r = -3$

$x = -3 - 2t - s + 3r.$

Svar:
$$\begin{cases} x = -3 - 2t - s + 3r \\ y = t \\ z = 1 - r \\ u = s \\ v = r \end{cases}$$

$$\begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + r \cdot \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Bestäm inversmatris:

Bara kvadratiska matriser kan ha invers. (om $\det \neq 0$)

$$A \cdot A^{-1} = I$$

$(A|I)$ totalmatris $\sim$
$\sim$ Gauss Jordan $\sim$

$\sim (I\,|\,A^{-1})$
sökt.

jfr.
$$A \cdot \bar{x} = B$$
$$(A|B) \sim \sim (I|\bar{x})$$

ex) $A = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -3 & 4 \\ 3 & -8 & 1 \end{pmatrix}$ $\quad A^{-1} = ?$

$$\left(\begin{array}{ccc|ccc} \multicolumn{3}{c|}{\overbrace{\quad\quad}^{A}} & \multicolumn{3}{c}{\overbrace{\quad\quad}^{I}} \\ 1 & -2 & 1 & 1 & 0 & 0 \\ 2 & -3 & 4 & 0 & 1 & 0 \\ 3 & -8 & 1 & 0 & 0 & 1 \end{array}\right) \begin{matrix} -2 & -3 \end{matrix} \sim \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & -2 & -2 & -3 & 0 & 1 \end{array}\right) 2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 2 & -7 & 2 & 1 \end{array}\right) \tfrac{1}{2} \sim \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7/2 & 1 & 1/2 \end{array}\right) \begin{matrix} -2 & -1 \end{matrix}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 9/2 & -1 & -1/2 \\ 0 & 1 & 0 & 5 & -1 & -1 \\ 0 & 0 & 1 & -7/2 & 1 & 1/2 \end{array}\right) 2 \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 29/2 & -3 & -5/2 \\ 0 & 1 & 0 & 5 & -1 & -1 \\ 0 & 0 & 1 & -7/2 & 1 & 1/2 \\ \multicolumn{3}{c|}{\underbrace{\quad\quad}_{I}} & \multicolumn{3}{c}{\underbrace{\quad\quad\quad}_{A^{-1}}} \end{array}\right)$$

$$A^{-1} = \frac{1}{2} \cdot \begin{pmatrix} 29 & -6 & -5 \\ 10 & -2 & -2 \\ -7 & 2 & 1 \end{pmatrix}$$

Koll: $A \cdot A^{-1} = I$

$$\frac{1}{2} \cdot \begin{pmatrix} 1 & -2 & 1 \\ 2 & -3 & 4 \\ 3 & -8 & 1 \end{pmatrix} \begin{pmatrix} 29 & -6 & -5 \\ 10 & -2 & -2 \\ -7 & 2 & 1 \end{pmatrix} = \frac{1}{2} \cdot \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

OK!

$$= \frac{1}{2} \begin{pmatrix} 1 \cdot 29 + (-2) \cdot 10 + 1 \cdot (-7) & 1 \cdot (-6) + (-2)(-2) + 1 \cdot 2 & 1 \cdot (-5) + (-2)(-2) + 1 \cdot 1 \\ 2 \cdot 29 + (-3) \cdot 10 + 4 \cdot (-7) & 2 \cdot (-6) + (-3)(-2) + 4 \cdot 2 & 2 \cdot (-5) + (-3)(-2) + 4 \cdot 1 \\ 3 \cdot 29 + (-8) \cdot 10 + 1 \cdot (-7) & 3(-6) + (-8) \cdot (-2) + 1 \cdot 2 & 3 \cdot (-5) + (-8) \cdot (-2) + 1 \cdot 1 \end{pmatrix} = I$$

$$AX = B$$

$$\underbrace{A^{-1} \cdot A}_{I} X = A^{-1} \cdot B$$

$$X = A^{-1} \cdot B$$

En entydig, unik lösning.

A måste vara kvadratisk.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Om A^{-1} existerar finns entydig lösning.
unik !

$\Updownarrow$

A^{-1} existerar $\iff$ $\det A \neq 0$

Entydig, unik lösning omm $\det A \neq 0$

Determinanter (kap 2.2 + delar av kap 6.)

är ett tal som hör till kvadratiska matriser

$\det(A) = |A|$

ex) $A = [\bar{a}\ \bar{b}] = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$

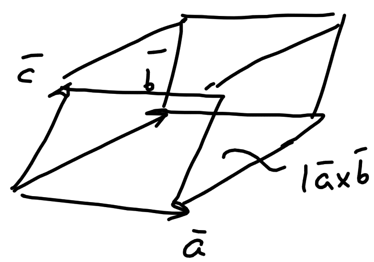
$\det A = \begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix} = 1\cdot 4 - 5\cdot 2 = -6$

(2x2) Arean $= |\det A| = |-6| = \underline{6}$ a.e.

$\det(A^T) = \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = 1\cdot 4 - 2\cdot 5 = -6$

$\therefore\ \det A = \det A^T$

(3x3) $B = [\bar{a}\ \bar{b}\ \bar{c}]$



Volym av parallellepiped

$V = |\det B|$ beloppet av determinanten.

även $V = ||\bar{a}\times\bar{b}|\cdot\bar{c}|$

$\det B = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}$

Sarrus regel (gäller ej större determinanter än 3x3.)

$= a_1\cdot b_2\, c_3 + b_1\cdot c_2\cdot a_3 + c_1\cdot a_2\cdot b_3 - a_3 b_2 c_1 - b_3 c_2 a_1 - c_3 a_2 b_1$

Räkneregler för determinanter

- någon Rad/kol $= 0 \Rightarrow \det = 0$

$$\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} = 0$$

- 2 Rad/kol lika $\Rightarrow \det = 0$

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 8 & 22 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 17 \end{vmatrix}$$ lika

- Triangulär matris : $\det = a_{11} \cdot a_{22} \cdots a_{nn}$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 0 & 0 & 2 \end{vmatrix} = 1 \cdot 4 \cdot 2 = 8 \qquad \det A = \prod_{i=1}^{n} a_{ii}$$

produkten av diagonal elementen.

- $\det A = \det A^T$

- $\det(AB) = \det A \cdot \det B$

$$\det(AA^{-1}) = \det I = 1 = \det A \cdot \det(A^{-1})$$

$$\therefore \det A^{-1} = \frac{1}{\det A}$$

- Byta plats på rad/kol $\Rightarrow$ determinanten byter tecken.

ex) $A = [\bar{a}\ \bar{b}] = \begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix}$

$|\det[\bar{a}\ \bar{b}]|$

Rad byte

$$\begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} = 5 \cdot 2 - 1 \cdot 4 = 6$$

$$\det A = \begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix} = 1 \cdot 4 - 5 \cdot 2 = -6.$$

- Bryta ut ett tal (k) ur en rad el. kolumn gör determinanten k ggr så stor. som den kvarvarande.

$$|B| = \begin{vmatrix} 10 & 2 \\ 50 & 4 \end{vmatrix} = 10 \cdot \begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix}$$

$10 \cdot \bar{a}$ $\quad \bar{b}$

$$= 10 \cdot (-6) = -60$$

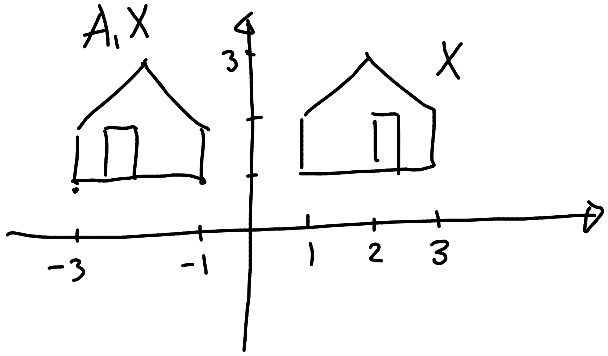
$$= 10 \cdot 2 \cdot \begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix} = 20 \cdot (1 \cdot 2 - 5 \cdot 1) = 20 \cdot (-3) = -60.$$

$$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$ speglar i y-axeln.

$$A_1 \cdot X = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x + 0 \cdot y \\ 0 \cdot x + 1 \cdot y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$



$$A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$A_5 = \begin{pmatrix} 0.5 & 1 \\ 1.5 & -1 \end{pmatrix}$$

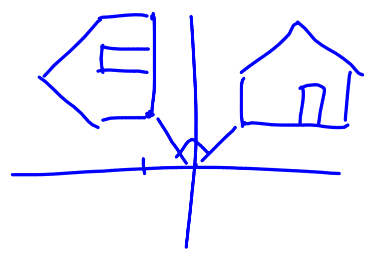
A_4

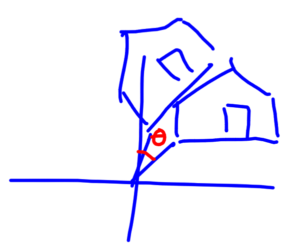
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

roterar 90°

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$





$$\underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\text{rotationsmatris}} \begin{pmatrix} x \\ y \end{pmatrix}$$

rotationsmatris
kring z-axeln
vinkeln θ.